***Proceedings*:**
National Conference on ' Information Technology: Setting Trends in Modern Era'
and 8th Annual Conference of Indian Society of Information Theory & Applications
March 18 – 20, 2006, N.C. College of Engineering, Israna (Panipat), Haryana, India
pp. 88 – 94

On a Sum Form Functional Equation and its Role in Information Theory

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Abstract. In this paper, the authors have obtained all general solutions of the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^{k} F(p_i) + \sum_{j=1}^{\ell} H(q_j) + \lambda \sum_{i=1}^{k} F(p_i) \sum_{j=1}^{\ell} H(q_j)$$

where $\lambda \neq 0$, $F : I \to \mathbb{R}$, $H : I \to \mathbb{R}$, $I = [0,1]$, $(p_1,\dots,p_k) \in \Gamma_k$, $(q_1,\dots,q_\ell) \in \Gamma_\ell$, $k \geq 3$, $\ell \geq 3$ being fixed integers and Γ_n denoting the set of all complete discrete probability distributions with n nonnegative components.

AMS Subject Classification (2000): 39B52, 39B82

Keywords: Sum form functional equation, additive function, multiplicative function.

1. Introduction

For $n = 1,2,3,\dots$; let

$$\Gamma_n = \left\{ (p_1,\dots,p_n) : p_i \geq 0, i = 1,\dots,n; \sum_{i=1}^{n} p_i = 1 \right\}$$

denote the set of all n-component complete discrete probability distributions with nonnegative elements. The axiomatic characterizations of the nonadditive entropies of degree α, $\alpha > 0$, $\alpha \neq 1$, given by Havrda and Charvát [4] and, the Shannon entropies [7] which are additive, lead to the study of the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^{k} F(p_i) + \sum_{j=1}^{\ell} F(q_j) + \lambda \sum_{i=1}^{k} F(p_i) \sum_{j=1}^{\ell} F(q_j) \tag{1.1}$$

where $(p_1,\dots,p_k) \in \Gamma_k$, $(q_1,\dots,q_\ell) \in \Gamma_\ell$, $\lambda \in \mathbb{R}$, $F : I \to \mathbb{R}$, $I = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $\mathbb{R}$ denoting the set of all real numbers.

If $\lambda = 0$, then (1.1) reduces to the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} F(p_i q_j) = \sum_{i=1}^{k} F(p_i) + \sum_{j=1}^{\ell} F(q_j) \tag{1.2}$$

due to Chaundy and McLeod [2]. This functional equation is useful in the characterization of the Shannon entropies. If $\lambda = 2^{1-\alpha} - 1$, $\alpha > 0$, $\alpha \neq 1$, then (1.1) is useful in characterizing the nonadditive entropies of degree α, $\alpha > 0$, $\alpha \neq 1$. See Behara and Nath [1].

Losonczi and Maksa [5] obtained the general solutions of (1.1) for all $(p_1,\dots,p_k) \in \Gamma_k$, $(q_1,\dots,q_\ell) \in \Gamma_\ell$ when $k \geq 3$ and $\ell \geq 3$ are fixed integers, in both cases namely when $\lambda = 0$ and $\lambda \neq 0$.

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There are several generalizations of (1.1) which contain only two unknown functions. One such generalization is

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}F(p_iq_j)=\sum_{i=1}^{k}F(p_i)+\sum_{j=1}^{\ell}H(q_j)+\lambda\sum_{i=1}^{k}F(p_i)\sum_{j=1}^{\ell}H(q_j) \tag{1.3}$$

where $F: I \to \mathbb{R}$ and $H: I \to \mathbb{R}$, $\lambda \in \mathbb{R}$.

The object of this paper is to investigate all general solutions of equation (1.3) for all $(p_1,\dots,p_k) \in \Gamma_k$, $(q_1,\dots,q_\ell) \in \Gamma_\ell$; $k \geq 3$, $\ell \geq 3$ being fixed integers and $\lambda \neq 0$.

The process of determining the general solutions of (1.3), when $\lambda \neq 0$, requires finding the general solutions of the multiplicative type functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}f(p_iq_j)=\sum_{i=1}^{k}f(p_i)\sum_{j=1}^{\ell}h(q_j) \tag{1.4}$$

in which $f: I \to \mathbb{R}$, $h: I \to \mathbb{R}$, $(p_1,\dots,p_k) \in \Gamma_k$, $(q_1,\dots,q_\ell) \in \Gamma_\ell$ and $k \geq 3$, $\ell \geq 3$ are fixed integers. This task has been accomplished in section 3 of this paper. The general solutions of equation (1.3), when $\lambda \neq 0$, have been obtained in section 4.

2. Some Preliminary Results

In this section, we mention some definitions and results which are needed in sections 3 and 4.

A mapping $a: I \to \mathbb{R}$ is said to be additive on the unit triangle

$$\Delta = \{(x,y): 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq x+y \leq 1\}$$

if it satisfies the functional equation $a(x+y) = a(x) + a(y)$ for all $(x,y) \in \Delta$. A mapping $A: \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation

$$A(x+y) = A(x) + A(y) \tag{2.1}$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

It is known (see Daróczy and Losonczi [3]) that every mapping $a: I \to \mathbb{R}$, additive on the unit triangle Δ, has a unique additive extension $A: \mathbb{R} \to \mathbb{R}$ in the sense that A satisfies (2.1) for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

Result 1 (see [5]). Let $\psi: I \to \mathbb{R}$ be a mapping which satisfies the equation

$$\sum_{i=1}^{n}\psi(p_i) = c$$

for all $(p_1,\dots,p_n) \in \Gamma_n$; c a given constant and $n \geq 3$ a fixed integer. Then there exists an additive mapping $a: \mathbb{R} \to \mathbb{R}$ such that

$$\psi(p) = a(p) - \frac{1}{n}a(1) + \frac{c}{n}$$

for all $p \in I$.

Definition 1. A mapping $m: I \to \mathbb{R}$ is said to be multiplicative on I if $m(0) = 0$, $m(1) = 1$ and $m(pq) = m(p)m(q)$ for all $p \in]0,1[$, $q \in]0,1[$ where $]0,1[= \{x \in \mathbb{R}: 0 < x < 1\}$.

Result 2 (see [6]). If a mapping $h: I \to \mathbb{R}$ satisfies the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}h(p_iq_j)=\sum_{i=1}^{k}h(p_i)\sum_{j=1}^{\ell}h(q_j)+(\ell-k)\,h(0)\sum_{j=1}^{\ell}h(q_j)+\ell(k-1)\,h(0) \tag{2.2}$$

for all $(p_1,\ldots,p_k)\in\Gamma_k$, $(q_1,\ldots,q_\ell)\in\Gamma_\ell$; $k\geq 3$, $\ell\geq 3$ being fixed integers, then h is of the form

$$h(p)=d(p)+h(0) \tag{2.3}$$

where $d:\mathbb{R}\to\mathbb{R}$ is an additive mapping such that

$$d(1)=\begin{cases}-\ell\, h(0) & \text{if } h(1)+(\ell-1)\,h(0)\neq 1\\ 1-\ell\, h(0) & \text{if } h(1)+(\ell-1)\,h(0)=1\end{cases} \tag{2.4}$$

or

$$h(p)=M(p)-e(p)+h(0) \tag{2.5}$$

where $e:\mathbb{R}\to\mathbb{R}$ is an additive mapping with

$$e(1)=\ell\, h(0) \tag{2.6}$$

and $M:I\to\mathbb{R}$ is a nonconstant nonadditive mapping which is multiplicative in the sense of Definition 1.

3. On the Functional Equation (1.4)

In this section, we prove the following:

Theorem 1. *Let $f:I\to\mathbb{R}$, $h:I\to\mathbb{R}$ be mappings which satisfy the functional equation* (1.4) *for all* $(p_1,\ldots,p_k)\in\Gamma_k$, $(q_1,\ldots,q_\ell)\in\Gamma_\ell$, $k\geq 3$, $\ell\geq 3$ *being fixed integers. Then, any general solution of* (1.4) *is of the form*

$$f(p)=b(p),\quad h \text{ an arbitrary real-valued mapping} \tag{3.1}$$

with $b(1)=0$ *or*

$$\left.\begin{aligned} f(p)&=b(p)-\frac{1}{k}\,b(1)-(\ell-1)\,f(0)[1-h(1)-(\ell-1)\,h(0)]^{-1},\quad f(0)\neq 0\\ h(p)&=a(p)-\frac{1}{\ell}\,a(1)+\frac{1}{\ell}\,[h(1)+(\ell-1)\,h(0)]\end{aligned}\right\} \tag{3.2}$$

with

$$\begin{aligned} b(1)\ &=\ -k\,f(0)\,[\ell-h(1)-(\ell-1)\,h(0)][1-h(1)-(\ell-1)\,h(0)]^{-1},\\ &\qquad (f(0)\neq 0 \text{ and } [1-h(1)-(\ell-1)\,h(0)]\neq 0)\end{aligned} \tag{3.3}$$

or

$$\left.\begin{aligned} f(p)&=f(1)\,d(p)+E(p),\ f(1)\neq 0\\ h(p)&=d(p)+h(0)\end{aligned}\right\} \tag{3.4}$$

with $d(1)=1-\ell\, h(0)$ *and* $E(1)=\ell\, f(1)\,h(0)$ *or*

$$\left.\begin{aligned} f(p)&=f(1)\,[M(p)-e(p)]+E(p),\ f(1)\neq 0\\ h(p)&=M(p)-e(p)+h(0)\end{aligned}\right\} \tag{3.5}$$

with $e(1)=\ell\, h(0)$ *and* $E(1)=\ell\, f(1)\,h(0)$; *where* $a:\mathbb{R}\to\mathbb{R}$, $b:\mathbb{R}\to\mathbb{R}$, $d:\mathbb{R}\to\mathbb{R}$, $e:\mathbb{R}\to\mathbb{R}$, $E:\mathbb{R}\to\mathbb{R}$ *are additive mappings and* $M:I\to\mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative in the sense of Definition* 1.

Proof. Let us put $q_1 = 1$, $q_2 = \ldots = q_\ell = 0$ in (1.4). We get

$$[h(1)+(\ell-1)h(0)-1]\sum_{i=1}^{k} f(p_i) = k(\ell-1)f(0) \tag{3.6}$$

for all $(p_1,\ldots,p_k)\in\Gamma_k$. Now, we divide our discussion into two cases:

Case 1. $\sum_{i=1}^{k} f(p_i)$ vanishes identically on Γ_k. Then (3.6) gives $f(0)=0$. Also

$$\sum_{i=1}^{k} f(p_i) = 0 \tag{3.7}$$

holds for all $(p_1,\ldots,p_k)\in\Gamma_k$. Hence, by Result 1, there exists an additive mapping $b:\mathbb{R}\to\mathbb{R}$ such that

$$f(p) = b(p) - \frac{1}{k}\,b(1) \tag{3.8}$$

for all $p\in I$. The substitution $p=0$, in (3.8), gives

$$b(1) = -k\,f(0). \tag{3.9}$$

Since $f(0)=0$, (3.9) gives $b(1)=0$. So, (3.8) reduces to

$$f(p) = b(p) \tag{3.10}$$

for all $p\in I$. From (1.4), (3.10) and the fact that $b(1)=0$, it follows that h is an arbitrary real-valued mapping. Thus, we have obtained the solution (3.1) of (1.4).

Case 2. $\sum_{i=1}^{k} f(p_i)$ does not vanish identically on Γ_k. In this case, there exists a probability distribution $(p_1^*,\ldots,p_k^*)\in\Gamma_k$ such that

$$\sum_{i=1}^{k} f(p_i^*) \neq 0. \tag{3.11}$$

Keeping in view (3.6), we further subdivide our discussion into two cases:

Case 2.1.

$$1-h(1)-(\ell-1)h(0) \neq 0. \tag{3.12}$$

Choosing $p_i = p_i^*$, $i=1,\ldots,k$ in (3.6), making use of (3.11) and (3.12), it follows that $f(0)\neq 0$ because $k\geq 3$ and $\ell\geq 3$ are fixed integers. Also, from (3.6) and (3.12), we get

$$\sum_{i=1}^{k} f(p_i) = k(\ell-1)f(0)\,[h(1)+(\ell-1)h(0)-1]^{-1} \tag{3.13}$$

with $f(0)\neq 0$. By Result 1, there exists an additive mapping $b:\mathbb{R}\to\mathbb{R}$ such that

$$f(p) = b(p) - \frac{1}{k}\,b(1) - (\ell-1)f(0)[1-h(1)-(\ell-1)h(0)]^{-1}, \quad f(0)\neq 0 \tag{3.14}$$

for all $p\in I$. The substitution $p=0$ in (3.14) gives $b(1)$ as mentioned in (3.3). Also, from (3.14); using the additivity of $b:\mathbb{R}\to\mathbb{R}$ and equation (3.3), it is easy to derive the equation

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} f(p_i q_j) = -k(\ell-1)f(0)\,[h(1)+(\ell-1)h(0)][1-h(1)-(\ell-1)h(0)]^{-1}, \quad (f(0)\neq 0) \tag{3.15}$$

valid for all $(p_1,\ldots,p_k)\in\Gamma_k$ and $(q_1,\ldots,q_\ell)\in\Gamma_\ell$. Making use of (3.13) and (3.15), equation (1.4) reduces

$$\sum_{j=1}^{\ell} h(q_j)=h(1)+(\ell-1)h(0)$$

for all $(q_1,\ldots,q_\ell)\in\Gamma_\ell$. By Result 1, there exists an additive mapping $a:\mathbb{R}\to\mathbb{R}$ such that

$$h(p)=a(p)-\frac{1}{\ell}\,a(1)+\frac{1}{\ell}\,[h(1)+(\ell-1)\,h(0)] \tag{3.16}$$

for all $p\in I$. Equations (3.14) and (3.16) constitute the solution (3.2), of (1.4), subject to the condition (3.3).

Case 2.2.

$$1-h(1)-(\ell-1)\,h(0)=0\,. \tag{3.17}$$

From (3.6) and (3.17), it follows that $f(0)=0$.

Let us put $p_1=1$, $p_2=\ldots=p_k=0$ in (1.4). We obtain

$$\sum_{j=1}^{\ell}\{f(q_j)-[f(1)+(k-1)\,f(0)]\,h(q_j)\}=-\ell(k-1)\,f(0)\,.$$

By Result 1, there exists a mapping $E:\mathbb{R}\to\mathbb{R}$ such that

$$f(p)=[f(1)+(k-1)\,f(0)]\,h(p)+E(p)-\frac{1}{\ell}\,E(1)-(k-1)\,f(0) \tag{3.18}$$

for all $p\in I$. Since $f(0)=0$, (3.18) reduces to

$$f(p)=f(1)\,h(p)+E(p)-\frac{1}{\ell}\,E(1)\,. \tag{3.19}$$

Substituting $p=0$ in (3.19) and using the fact that $f(0)=0=E(0)$, we obtain

$$E(1)=\ell\,f(1)\,h(0)\,. \tag{3.20}$$

From (3.19) and (3.20), one can derive the equations (using the additivity of E)

$$\sum_{i=1}^{k} f(p_i)=f(1)\sum_{i=1}^{k} h(p_i)+(\ell-k)\,f(1)\,h(0) \tag{3.21}$$

and

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell} f(p_iq_j)=f(1)\sum_{i=1}^{k}\sum_{j=1}^{\ell} h(p_iq_j)+\ell(1-k)\,f(1)\,h(0) \tag{3.22}$$

for all $(p_1,\ldots,p_k)\in\Gamma_k$ and $(q_1,\ldots,q_\ell)\in\Gamma_\ell$. Now, from (1.4), (3.21) and (3.22), one can derive the equation

$$f(1)\sum_{i=1}^{k}\sum_{j=1}^{\ell} h(p_iq_j)=f(1)\sum_{i=1}^{k} h(p_i)\sum_{j=1}^{\ell} h(q_j)+(\ell-k)\,f(1)\,h(0)\sum_{j=1}^{\ell} h(q_j)+\ell(k-1)\,f(1)\,h(0) \tag{3.23}$$

for all $(p_1,\ldots,p_k)\in\Gamma_k$ and $(q_1,\ldots,q_\ell)\in\Gamma_\ell$.

We prove that $f(1)\neq 0$. If possible, suppose $f(1)=0$. Then, from (3.19) and (3.20), it follows that

$$f(p)=E(p) \tag{3.24}$$

for all $p\in I$ with

$$E(1)=0\,. \tag{3.25}$$

Now, making use of (3.11), (3.24), (3.25) and the additivity of E, we have

$$0 \neq \sum_{i=1}^{k} f(p_i^*) = \sum_{i=1}^{k} E(p_i^*) = E(1) = 0$$

a contradiction. So, the only possibility is that $f(1) \neq 0$. Now, equation (3.23) gives (2.2) for all $(p_1, \ldots, p_k) \in \Gamma_k$ and $(q_1, \ldots, q_\ell) \in \Gamma_\ell$.

Since $1 - h(1) - (\ell - 1)h(0) = 0$, by Result 2, it follows that h is of the form (2.3) in which $d : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $d(1) = 1 - \ell\, h(0)$ or h is of the form (2.5) in which $e : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $e(1) = \ell\, h(0)$.

From (3.19), (3.20) and (2.3), we get the solution (3.4) with $d(1) = 1 - \ell\, h(0)$ and $E(1) = \ell\, f(1)\, h(0)$, $f(1) \neq 0$.

From (3.19), (3.20) and (2.5), we get the solution (3.5) with $e(1) = \ell\, h(0)$ and $E(1) = \ell\, f(1)\, h(0)$, $f(1) \neq 0$. This completes the proof of Theorem 1.

4. The General Solutions of (1.3) when $\lambda \neq 0$

We prove:

Theorem 2. *Let $F : I \to \mathbb{R}$, $H : I \to \mathbb{R}$ be mappings which satisfy the functional equation* (1.3) *for all* $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_\ell) \in \Gamma_\ell$, $k \geq 3$, $\ell \geq 3$ *being fixed integers and* $\lambda \neq 0$. *Then any general solution of* (1.3) *is of the form*

$$F(p) = \frac{1}{\lambda}\{b(p) - p\}; \ \ H \text{ an arbitrary real-valued mapping} \tag{4.1}$$

with $b(1) = 0$ *or*

$$\left.\begin{aligned} F(p) &= \frac{1}{\lambda}\left\{b(p) - \frac{1}{k}\, b(1) + (\ell - 1) F(0)[H(1) + (\ell - 1)H(0)]^{-1} - p\right\}, \quad F(0) \neq 0 \\ H(p) &= \frac{1}{\lambda}\left\{a(p) - \frac{1}{\ell}\, a(1) + \frac{1}{\ell}\,[\lambda(H(1) + (\ell - 1)H(0)) + 1] - p\right\} \end{aligned}\right\} \tag{4.2}$$

with

$$\begin{aligned} b(1) \;=\; & k\, F(0)\,[\ell - 1 - \lambda(H(1) + (\ell - 1)H(0))][H(1) + (\ell - 1)H(0)]^{-1}, \\ & (F(0) \neq 0, \ [H(1) + (\ell - 1)H(0)] \neq 0) \end{aligned} \tag{4.3}$$

or

$$\left.\begin{aligned} F(p) &= \frac{1}{\lambda}\{[\lambda F(1) + 1]\, d(p) + E(p) - p\}, \ \ \lambda F(1) + 1 \neq 0 \\ H(p) &= \frac{1}{\lambda}\{d(p) + \lambda H(0) - p\} \end{aligned}\right\} \tag{4.4}$$

with $d(1) = 1 - \lambda\ell\, H(0)$ *and* $E(1) = \lambda\ell\,[\lambda F(1) + 1]\, H(0)$ *or*

$$\left.\begin{aligned} F(p) &= \frac{1}{\lambda}\{[\lambda F(1) + 1]\,[M(p) - e(p)] + E(p) - p\}, \quad \lambda F(1) + 1 \neq 0 \\ H(p) &= \frac{1}{\lambda}\{M(p) - e(p) + \lambda H(0) - p\} \end{aligned}\right\} \tag{4.5}$$

with $e(1) = \lambda\ell\, H(0)$ *and* $E(1) = \lambda\ell\,[\lambda F(1) + 1]\, H(0)$; *where* $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $d : \mathbb{R} \to \mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$, $E : \mathbb{R} \to \mathbb{R}$ *are additive mappings and* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative in the sense of Definition* 1.

Proof. Let us write (1.3) in the form

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}[\lambda F(p_i q_j) + p_i q_j] = \sum_{i=1}^{k}[\lambda F(p_i) + p_i]\sum_{j=1}^{\ell}[\lambda H(q_j) + q_j]. \tag{4.6}$$

Define the mappings $f: I \to \mathbb{R}$, $h: I \to \mathbb{R}$ as

$$\left.\begin{aligned} f(x) &= \lambda F(x) + x \\ h(x) &= \lambda H(x) + x \end{aligned}\right\} \tag{4.7}$$

for all $x \in I$. Then, (4.6) reduces to the functional equation (1.4). Now Theorem 2 follows from (4.7) and Theorem 1. The details are omitted.

Acknowledgement

The authors are grateful to the referee for his valuable suggestions.

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