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Some Sum Form Functional Equations Containing at Most Two Unknown Mappings

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Abstract. The general solutions of sum form functional equations have been investigated.
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1. Introduction

For $n = 1,2,\ldots$, let $\Gamma_n = \left\{(p_1,\ldots p_n) : p_i \geq 0, i = 1,\ldots,n; \sum_{i=1}^{n} p_i = 1\right\}$ denote the set of all complete finite discrete n-component probability distributions with nonnegative elements. The functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) \tag{1.1}$$

with $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$, $f : I \to \mathbb{R}$, $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $\mathbb{R}$ denoting the set of all real numbers, plays an important role in the characterization of the Shannon [19] entropies defined as

$$H_n(p_1,\ldots,p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{1.2}$$

where $H_n : \Gamma_n \to \mathbb{R}$, $n = 1,2\ldots$ and $0\log_2 0 := 0$. The functional equation (1.1) was first studied by Chaundy and Mcleod [3] who came across it while studying some problems in statistical thermodynamics. For some research work concerning (1.1), see [1], [4], [5] and [12].

A generalization of (1.1) is the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} g(q_j) + \sum_{j=1}^{m} f(q_j) \tag{1.3}$$

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in which $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$, $(p_1,\dots,p_n) \in \Gamma_n$ and $(q_1,\dots,q_m) \in \Gamma_m$. If $g(x) = x$ for all $x \in I$, then (1.3) reduces to (1.1). If $g(x) = \lambda f(x) + x$ for all $x \in I$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ being a parameter, then (1.3) reduces to the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \tag{1.4}$$

where $f : I \to \mathbb{R}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$. The functional equation (1.4) is useful in characterizing the nonadditive entropies

$$H_n^{\alpha}(p_1,\dots,p_n) = (1-2^{1-\alpha})^{-1}\Big(1 - \sum_{i=1}^{n} p_i^{\alpha}\Big) \tag{1.5}$$

where $H_n^{\alpha} : \Gamma_n \to \mathbb{R}$, $n = 1,2,\dots$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\alpha \neq 1$, $0^{\alpha} := 0$ and $1^{\alpha} := 1$. The nonadditive entropies $H_n^{\alpha}, n = 1,2\dots$, defined above, are due to Havrda and Charvát [7]. For some research work concerning (1.4) see [2], [8], [9], [10], [11], [13]. Indeed, the entropies (1.5) arise when (1.4) is considered by taking $\lambda = 2^{1-\alpha} - 1$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\alpha \neq 1$, $0^{\alpha} := 0, 1^{\alpha} := 1$ and imposing some regularity condition on f.

Nath [14] determined the Lebesgue measurable solutions of (1.3) in two cases: (i) for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$, $n = 1,2,\dots$; $m = 1,2,\dots$ (ii) for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ fixed integers.

In addition to the functional equations (1.3) and (1.4), we mention below the following functional equations:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) + m(n-1)g(0) \tag{1.6}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) + m(n-1)f(0) \tag{1.7}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} h(q_j) \tag{1.8}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} H(p_i) + \sum_{j=1}^{m} H(q_j) + \lambda \sum_{i=1}^{n} H(p_i) \sum_{j=1}^{m} H(q_j). \tag{1.9}$$

In equations (1.6) to (1.9), the mappings g, f, h, F and H are real-valued mappings, each with domain I.

The object of this paper is to determine the general solutions of each of the functional equations (1.3), (1.6), (1.7), (1.8) and (1.9) valid for all $(p_1,\dots,p_n) \in \Gamma_n$, $(q_1,\dots,q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers.

The functional equation (1.9) is a generalization of (1.4). It contains two unknown mappings. Recently, the authors [17] have discussed the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} F(p_i) + \sum_{j=1}^{m} H(q_j) + \lambda \sum_{i=1}^{n} F(p_i) \sum_{j=1}^{m} H(q_j)$$

which also contains two unknown mappings and is a generalization of (1.4).

2. Preliminary Results

In this section, we mention some definitions and results needed for the development of sections 3, 4 and 5.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I if it satisfies the equation $a(x+y) = a(x)+a(y)$ for all $(x, y) \in \Delta$ where $\Delta = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1, 0 \le x+y \le 1\}$. It is known (see [1]) that if a mapping $a : I \to \mathbb{R}$ is additive on I, then it has a unique additive extension $A : \mathbb{R} \to \mathbb{R}$ in the sense that $A(x+y) = A(x)+A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $m : I \to \mathbb{R}$ is said to be multiplicative on I if $m(0) = 0$, $m(1) = 1$ and $m(pq) = m(p)m(q)$ holds for all $p \in]0,1[$, $q \in]0,1[$ where $]0,1[= \{x \in \mathbb{R} : 0 < x < 1\}$.

Result 1 (see [11]). *Suppose a mapping $\varphi : I \to \mathbb{R}$ satisfies the functional equation*

$$\sum_{i=1}^{n} \phi(p_i) = c$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \ge 3$ a fixed integer and c a given constant. Then there exists a mapping $\Psi : \mathbb{R} \to \mathbb{R}$, additive on $\mathbb{R}$, such that

$$\phi(p) = \Psi(p) - \frac{1}{n}\Psi(1) + \frac{c}{n}$$

for all $p \in I$.

Result 2 (see [11]). *If a mapping $f : I \to \mathbb{R}$ satisfies the equation* (1.1) *for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $n \ge 3$, $m \ge 3$ being fixed integers, then it is of the form*

$$f(p) = \begin{cases} c + c(nm-n-m)p + a_1(p) + D(p, p) & \text{if } 0 < p \le 1 \\ c & \text{if } p = 0 \end{cases} \tag{2.1}$$

where $c = f(0)$ an arbitrary real constant; $a_1 : \mathbb{R} \to \mathbb{R}$ is additive; $D : \mathbb{R}\times\,]0,1] \to \mathbb{R}$ is additive in the first variable; and there exists a mapping $E : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ additive in both variables such that $a_1(1) = E(1,1)$ and

$$D(pq, pq) - D(pq, p) - D(pq, q) = E(p, q) \tag{2.2}$$

holds for all $p \in]0,1]$, $q \in]0,1]$, $]0,1] = \{x \in \mathbb{R} : 0 < x \le 1\}$.

From (2.2), it can be easily deduced that

$$a_1(1) + D(1,1) = 0 \tag{2.3}$$

as $E(1,1) = a_1(1)$.

Result 3 (see [16]). *If a mapping $g : I \to \mathbb{R}$ satisfies the functional equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} g(q_j) + (m-n)g(0) \sum_{j=1}^{m} g(q_j) + m(n-1)g(0) \tag{2.4}$$

for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$, $n\ge 3$, $m\ge 3$ *being fixed integers, then* g *is of the form*

$$g(p)=b_1(p)+g(0),\quad 0\le p\le 1 \tag{2.5}$$

where $b_1:\mathbb{R}\to\mathbb{R}$ *is an additive mapping with*

$$b_1(1)=\begin{cases}-mg(0) & if\ \ g(1)+(m-1)g(0)\neq 1\\ 1-mg(0) & if\ \ g(1)+(m-1)g(0)=1\end{cases} \tag{2.6}$$

or

$$g(p)=M(p)-b(p)+g(0),\quad 0\le p\le 1 \tag{2.7}$$

where $b:\mathbb{R}\to\mathbb{R}$ *is an additive mapping with* $b(1)=mg(0)$ *and* $M:I\to\mathbb{R}$ *is a nonconstant nonadditive mapping which is also multiplicative on* I.

Result 4 (see [18]). *If a mapping* $f:I\to\mathbb{R}$ *satisfies the functional equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_iq_j)=\sum_{j=1}^{m} f(q_j) \tag{2.8}$$

for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$, $n\ge 3$, $m\ge 3$ *being fixed integers then there exists an additive mapping* $a_2:\mathbb{R}\to\mathbb{R}$ *such that* $f(p)=a_2(p)$ *for all* $p\in I$.

3. The General Solutions of (1.3)

The main result of this section is:

Theorem 1. *Suppose the mappings* $f:I\to\mathbb{R}$ *and* $g:I\to\mathbb{R}$ *satisfy* (1.3) *for all* $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$, $n\ge 3$, $m\ge 3$ *being fixed integers. Then any general solution* (f,g) *of* (1.3) *is of the form*

$$\begin{cases} f(p)=\begin{cases} c+c(nm-n-m)p+a_1(p)+D(p,p) & if\ \ 0<p\le 1\\ c & if\ \ p=0\end{cases}\\ g(p)=b_1(p)+\dfrac{1}{m}(1-b_1(1))\end{cases} \tag{3.1}$$

with c,a_1,D *as explained in* Result 2 *or*

$$\begin{cases} f(p)=a_2(p)-\dfrac{1}{n}a_2(1)+\dfrac{1}{n}[f(1)+(n-1)f(0)]\\ g(p)=b_1(p)-\dfrac{1}{m}b_1(1)+(n-1)f(0)[f(1)+(n-1)f(0)]^{-1}\end{cases} \tag{3.2}$$

with $a_2(1)=f(1)-f(0)$ *and* $f(1)+(n-1)f(0)\neq 0$ *or*

$$f(p)=a_2(p),\quad g\ arbitrary \tag{3.3}$$

with $a_2(1) = 0$ *or*

$$f(p) = -a_3(1)M(p) + a_3(p), \quad g(p) = M(p) - b(p) + \frac{1}{m}b(1) \tag{3.4}$$

where $a_3(1) \neq 0$; $a_i : \mathbb{R} \to \mathbb{R}$ $(i = 1,2,3)$; $b_1 : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$ *are additive mappings and* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative on* I.

Proof. We divide our discussion into two cases:

Case 1. $\sum_{j=1}^{m} g(q_j) - 1$ vanishes identically on Γ_m.

In this case, $\sum_{j=1}^{m} g(q_j) = 1$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 1, there exists an additive mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that

$$g(p) = b_1(p) + \frac{1}{m}(1 - b_1(1)), \quad 0 \leq p \leq 1. \tag{3.5}$$

Substituting $\sum_{j=1}^{m} g(q_j) = 1$ in (1.3), (1.1) follows. So, f is of the form (2.1) with c, a_1 and D as explained in Result 2. Equations (2.1) and (3.5), taken together, constitute the solution (3.1) of (1.3).

Case 2. $\sum_{j=1}^{m} g(q_j) - 1$ does not vanish identically on Γ_m.

Let us put $p_1 = 1, p_2 = \ldots = p_n = 0$ in (1.3). We get

$$[f(1) + (n-1)f(0)] \sum_{j=1}^{m} g(q_j) = m(n-1)f(0) \tag{3.6}$$

valid for all $(q_1, \ldots q_m) \in \Gamma_m$.

Case 2.1 $f(1) + (n-1)f(0) \neq 0$.

In this case, (3.6) gives

$$\sum_{j=1}^{m} g(q_j) = m(n-1)f(0)[f(1) + (n-1)f(0)]^{-1}. \tag{3.7}$$

By Result 1, there exists an additive mapping $b_1 : \mathbb{R} \to \mathbb{R}$ such that

$$g(p) = b_1(p) - \frac{1}{m}b_1(1) + (n-1)f(0)[f(1) + (n-1)f(0)]^{-1} \tag{3.8}$$

for all $p \in I$. Equations (1.3) and (3.7) give

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = m(n-1)f(0)[f(1) + (n-1)f(0)]^{-1} \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j)$$

from which it follows that

$$[f(1) - (n-1)(m-1)f(0)][f(1) + (n-1)f(0)]^{-1} \sum_{i=1}^{n} f(p_i) = f(1) - (n-1)(m-1)f(0). \tag{3.9}$$

Also, from (3.7),

$$\sum_{j=1}^{m} g(q_j) - 1 = -[f(1)-(n-1)(m-1)f(0)][f(1)+(n-1)f(0)]^{-1}. \tag{3.10}$$

Since $\sum_{j=1}^{m} g(q_j) - 1$ does not vanish identically on Γ_m, it follows from (3.10) that $f(1)-(n-1)(m-1)f(0)$ is nonzero. Consequently, (3.9) reduces to the equation

$$\sum_{i=1}^{n} f(p_i) = f(1) + (n-1)f(0)$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ a fixed integer. By Result 1, there exists an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = a_2(p) - \frac{1}{n}a_2(1) + \frac{1}{n}[f(1)+(n-1)f(0)] \tag{3.11}$$

for all $p \in I$ with $a_2(1) = f(1) - f(0)$. Equations (3.11) and (3.8), together with the condition $f(1) + (n-1)f(0) \neq 0$, constitute the solution (3.2) of (1.3).

Case 2.2

$$f(1) + (n-1)f(0) = 0. \tag{3.12}$$

From (3.6) and (3.12), it follows that $f(0) = f(1) = 0$. Let us write (1.3) in the form

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{m} f(p_i q_j) - f(p_i) \sum_{j=1}^{m} g(q_j) - p_i \sum_{j=1}^{m} f(q_j) \right] = 0.$$

By Result 1, there exists a mapping $A_1 : \mathbb{R} \times \Gamma_m \to \mathbb{R}$, additive in the first variable, such that

$$\sum_{j=1}^{m} f(p q_j) - f(p) \sum_{j=1}^{m} g(q_j) - p \sum_{j=1}^{m} f(q_j) = A_1(p; q_1, \ldots, q_m) \tag{3.13}$$

holds for all $p \in I$ and all $(q_1, \ldots, q_m) \in \Gamma_m$ with $A_1(1; q_1, \ldots q_m) = 0$ as $f(0) = f(1) = 0$. Let $(r_1, \ldots r_m) \in \Gamma_m$ be any probability distribution and $x \in I$. Putting $p = xr_t$, $t = 1, \ldots, m$ successively in (3.13); adding the resulting m equations so obtained; using the additivity of A_1 in the first variable; using the value of $\sum_{t=1}^{m} f(xr_t)$ obtained from (3.13) by replacing p by x and q_j by r_j; and performing necessary calculations, it follows that

$$\begin{aligned} &\sum_{t=1}^{m} \sum_{j=1}^{m} f(xr_t q_j) - f(x) \sum_{t=1}^{m} g(r_t) \sum_{j=1}^{m} g(q_j) \\ &= x \sum_{t=1}^{m} f(r_t) \sum_{j=1}^{m} g(q_j) + A_1(x; r_1, \ldots r_m) \sum_{j=1}^{m} g(q_j) + x \sum_{j=1}^{m} f(q_j) + A_1(x; q_1, \ldots, q_m). \end{aligned} \tag{3.14}$$

The left hand side of (3.14) is symmetric in r_t and q_j, $t = 1, \ldots, m$; $j = 1, \ldots, m$. Hence the right hand side of (3.14) must also be symmetric in r_t and q_j, $t = 1, \ldots, m$; $j = 1, \ldots, m$.

This gives rise to the symmetry equation

$$\Big[A_1(x;q_1;\ldots,q_m)+x\sum_{j=1}^{m}f(q_j)\Big]\Big[\sum_{t=1}^{m}g(r_t)-1\Big] \tag{3.15}$$
$$=\Big[A_1(x;r_1,\ldots,r_m)+x\sum_{t=1}^{m}f(r_t)\Big]\Big[\sum_{j=1}^{m}g(q_j)-1\Big].$$

Since $\sum_{j=1}^{m}g(q_j)-1$ does not vanish identically on Γ_m, there exists a probability distribution $(r_1^*,\ldots,r_m^*)\in\Gamma_m$ such that $\sum_{t=1}^{m}g(r_t^*)-1\neq 0$. Replacing r_t by r_t^* in (3.15), $t=1,\ldots,m$ and making use of the fact that $\sum_{t=1}^{m}g(r_t^*)-1\neq 0$, it follows that

$$A_1(x;q_1,\ldots,q_m)=B(x)\Big[\sum_{j=1}^{m}g(q_j)-1\Big]-x\sum_{j=1}^{m}f(q_j) \tag{3.16}$$

where $B:\mathbb{R}\to\mathbb{R}$ is defined as

$$B(x)=\Big[\sum_{t=1}^{m}g(r_t^*)-1\Big]^{-1}\Big[A_1(x;r_1^*,\ldots,r_m^*)+x\sum_{t=1}^{m}f(r_t^*)\Big] \tag{3.17}$$

for all $x\in\mathbb{R}$. Since A_1 is additive in the first variable, so $B:\mathbb{R}\to\mathbb{R}$ is also additive. If we put $x=1$ in (3.16) and use the fact that $A_1(1,q_1,\ldots q_m)=0$, we obtain the equation

$$B(1)\Big[\sum_{j=1}^{m}g(q_j)-1\Big]=\sum_{j=1}^{m}f(q_j)$$

valid for all $(q_1,\ldots,q_m)\in\Gamma_m$, $m\geq 3$ being a fixed integer.

Case 2.2.1 $B(1)=0$.

In this case, (3.17) gives the equation $\sum_{j=1}^{m}f(q_j)=0$. By Result 1, there exists an additive mapping $a_2:\mathbb{R}\to\mathbb{R}$ such that

$$f(p)=a_2(p)-\frac{1}{m}a_2(1) \tag{3.18}$$

holds for all $p\in I$. Putting $p=0$ in (3.18) and using $f(0)=0$, $a_2(0)=0$, it follows that

$$f(p)=a_2(p) \tag{3.19}$$

for all $p\in I$. Since $f(1)=0$, so $a_2(1)=0$. Now, from (3.19), the additivity of $a_2:\mathbb{R}\to\mathbb{R}$ and the fact that $a_2(1)=0$, it can be easily concluded that each of the three summands $\sum_{i=1}^{n}f(p_i)$, $\sum_{j=1}^{m}f(q_j)$ and $\sum_{i=1}^{n}\sum_{j=1}^{m}f(p_iq_j)$ equals zero for all $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$, $n\geq 3$, $m\geq 3$ being fixed integers. Consequently, now, it follows from (1.3) that $g:I\to\mathbb{R}$ is, indeed, an arbitrary mapping. Equation (3.19), together with $a_2(1)=0$ and the fact that $g:I\to\mathbb{R}$ is an arbitrary mapping, constitutes the solution (3.3) of (1.3).

Case 2.2.2 $B(1) \neq 0$.

In this case, let us write (3.17) in the form

$$\sum_{j=1}^{m}[f(q_j)-B(1)g(q_j)+B(1)q_j]=0.$$

By Result 1, there exists a mapping $\bar{A}:\mathbb{R}\to\mathbb{R}$ such that

$$f(p)-B(1)g(p)+B(1)p=\bar{A}(p)-\frac{1}{m}\bar{A}(1) \tag{3.20}$$

with

$$\bar{A}(1)=mB(1)g(0) \tag{3.21}$$

as $f(0)=0, \bar{A}(0)=0$. From (3.20) and (3.21), it follows that

$$f(p)=\bar{A}(p)+B(1)[g(p)-p-g(0)] \tag{3.22}$$

for all $p\in I$. From (3.22) and the additivity of $\bar{A}$, the following three equations follow:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}f(p_iq_j) = \bar{A}(1)+B(1)\Big[\sum_{i=1}^{n}\sum_{j=1}^{m}g(p_iq_j)-1-nmg(0)\Big] \tag{3.23}$$

$$\sum_{i=1}^{n}f(p_i) = \bar{A}(1)+B(1)\Big[\sum_{i=1}^{n}g(p_i)-1-ng(0)\Big] \tag{3.24}$$

$$\sum_{j=1}^{m}f(q_j) = \bar{A}(1)+B(1)\Big[\sum_{j=1}^{m}g(q_j)-1-mg(0)\Big]. \tag{3.25}$$

From equations (1.3), (3.23), (3.24), (3.25), (3.21) and the fact that $B(1)\neq 0$ equation (2.4) follows. Hence, by Result 3, g is of the form (2.5) in which $b_1:\mathbb{R}\to\mathbb{R}$ is an additive mapping with $b_1(1)$ given by (2.6) or g is of the form (2.7) in which $b:\mathbb{R}\to\mathbb{R}$ is an additive mapping with $b(1)=mg(0)$ and $M:I\to\mathbb{R}$ is a nonconstant nonadditive multiplicative mapping. From (2.5), (2.6) and the additivity of $b_1:\mathbb{R}\to\mathbb{R}$, we obtain

$$\sum_{j=1}^{m}g(q_j)=\begin{cases}0 & \text{if } g(1)+(m-1)g(0)\neq 1\\ 1 & \text{if } g(1)+(m-1)g(0)=1\end{cases}$$

valid for all $(q_1,\ldots,q_m)\in\Gamma_m$, $m\geq 3$ being a fixed integer. Since $\sum_{j=1}^{m}g(q_j)-1$ does not vanish identically on Γ_m, so the only possibility is that $\sum_{j=1}^{m}g(q_j)=0$ as the condition $g(1)+(m-1)g(0)\neq 1$ ensures that $\sum_{j=1}^{m}g(q_j)-1$ does not vanish identically on Γ_m. Then, by Result 1, there exists an additive mapping $b_1:\mathbb{R}\to\mathbb{R}$ such that

$$g(p)=b_1(p)-\frac{1}{m}b_1(1) \tag{3.26}$$

for all $p \in I$. Also, if we put $\sum_{j=1}^{m} g(q_j) = 0$ in (1.3), we get the equation (2.8). By Result 4, it follows that f is of the form (3.19). The solution consisting of (3.19) (with $a_2(1) = 0$) and (3.26), is included in (3.3).

From (2.7) and (3.22), we obtain

$$f(p) = B(1)M(p) + [\bar{A}(p) - B(1)b(p) - B(1)p] \tag{3.27}$$

for all $p \in I$. Define $a_3 : \mathbb{R} \to \mathbb{R}$ as

$$a_3(x) = \bar{A}(x) - B(1)b(x) - B(1)x \tag{3.28}$$

for all $x \in \mathbb{R}$. Then $a_3 : \mathbb{R} \to \mathbb{R}$ is additive. Making use of (3.28), (3.21) and the fact $b(1) = mg(0)$, it can be easily seen that $a_3(1) = -B(1) \neq 0$. Hence

$$f(p) = -a_3(1)M(p) + a_3(p) \tag{3.29}$$

where $a_3(1) \neq 0$. Equations (3.29) and (2.7), along with the condition $a_3(1) \neq 0$, constitute the solution (3.4) of(1.3). □

If we omit the second term appearing on the right hand side of equation (2.4), we obtain (1.6). This functional equation (1.6) is useful in obtaining the required general solutions of equations (1.7), (1.8) and (1.9). We accomplish this task in sections 4 and 5.

4. The Functional Equations (1.6) and (1.7)

We prove:

Theorem 2. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $g : I \to \mathbb{R}$ be a mapping which satisfies the functional equation* (1.6) *for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. Then g is of the form* (*for all $p \in I$*)

$$g(p) = a(p) + g(0) \tag{4.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is additive with

$$a(1) = \begin{cases} -mg(0) & \text{if } g(1) + (n-1)g(0) \neq 1 \\ 1 - ng(0) & \text{if } g(1) + (n-1)g(0) = 1 \end{cases} \tag{4.2}$$

or

$$g(p) = M(p) - b(p) + g(0) \tag{4.3}$$

with

$$(n-m)g(0) = 0 \tag{4.4}$$

where $b : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $b(1) = mg(0)$ and $M : I \to \mathbb{R}$ is a nonconstant nonadditive mapping which is multiplicative on I.

Proof. Let us put $p_1 = 1, p_2 = \ldots = p_n = 0$ in (1.6). We obtain

$$[g(1)+(n-1)g(0)-1]\sum_{j=1}^{m} g(q_j) = 0 \tag{4.5}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$.

Case 1. $g(1)+(n-1)g(0) \neq 1$.

In this case, we obtain $\sum_{j=1}^{m} g(q_j) = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 1, it follows that g is of the form (4.1) with $a(1) = -mg(0)$.

Case 2.

$$g(1)+(n-1)g(0) = 1. \tag{4.6}$$

In this case, let us write (1.6) in the form

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{n} g(p_i q_j) - g(q_j)\sum_{i=1}^{n} g(p_i)\right\} = m(n-1)g(0).$$

By Result 1, it follows that there exists a mapping $A_1 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$\sum_{i=1}^{n} g(p_i q) - g(q)\sum_{i=1}^{n} g(p_i) = A_1(p_1, \ldots, p_n; q) - g(0)\left[\sum_{i=1}^{n} g(p_i) - n\right] \tag{4.7}$$

for all $q \in I$ with

$$A_1(p_1, \ldots, p_n; 1) = mg(0)\left[\sum_{i=1}^{n} g(p_i) - 1\right]. \tag{4.8}$$

Now proceeding exactly as in Case 2.2 of Theorem 1, the symmetry equation

$$A_1(p_1, \ldots p_n; x)\left[\sum_{t=1}^{n} g(r_t) - 1\right] = A_1(r_1, \ldots, r_n; x)\left[\sum_{i=1}^{n} g(p_i) - 1\right] \tag{4.9}$$

valid for all $(r_1, \ldots, r_n) \in \Gamma_n$, $(p_1, \ldots, p_n) \in \Gamma_n$ and $x \in I$ can be obtained.

Case 2.1 $\sum_{t=1}^{n} g(r_t) - 1$ vanishes identically on Γ_n.

In this case, by Result 1, it follows that g is of the form (4.1) with $a(1) = 1 - ng(0)$.

Case 2.2 $\sum_{t=1}^{n} g(r_t) - 1$ does not vanish identically on Γ_n.

In this case, there exists a probability distribution $(r_1^*,\ldots,r_n^*)\in\Gamma_n$ such that $\sum_{t=1}^{n} g(r_t^*)-1\neq 0$. Now, from (4.9), it follows that

$$A_1(p_1,\ldots,p_n;x)=b(x)\left[\sum_{i=1}^{n} g(p_i)-1\right] \tag{4.10}$$

where $b:\mathbb{R}\to\mathbb{R}$ is defined as $b(x)=A_1(r_1^*,\ldots,r_n^*;x)\left[\sum_{t=1}^{n} g(r_t^*)-1\right]^{-1}$ for all $x\in\mathbb{R}$. Clearly, $b:\mathbb{R}\to\mathbb{R}$ is additive and making use of (4.8), it can also be proved that

$$b(1)=mg(0). \tag{4.11}$$

From (4.7), (4.10) and the additivity of $b:\mathbb{R}\to\mathbb{R}$, the equation

$$\sum_{i=1}^{n}[g(p_iq)-b(p_iq)-g(0)]-[g(q)+b(q)-g(0)]\sum_{i=1}^{n} g(p_i)=0 \tag{4.12}$$

follows. Let us define a mapping $M:I\to\mathbb{R}$ as

$$M(p)=g(p)+b(p)-g(0) \tag{4.13}$$

for all $p\in I$. Then $M(0)=0$. Also, from (4.13), (4.12) and (4.11), it follows that

$$\sum_{i=1}^{n} M(p_iq)-M(q)\sum_{i=1}^{n} M(p_i)+(m-n)g(0)M(q)=0 \tag{4.14}$$

valid for all $(p_1,\ldots,p_n)\in\Gamma_n$ and $q\in I$. The substitution $q=1$ in (4.14) gives

$$[1-M(1)]\sum_{i=1}^{n} M(p_i)=(n-m)g(0)M(1). \tag{4.15}$$

Now, we prove that (4.4) holds. To the contrary, suppose $(n-m)g(0)\neq 0$. Let us put $p=1$ in (4.13) and then make use of (4.6) and (4.11). We obtain

$$M(1)=1+(m-n)g(0). \tag{4.16}$$

Now, from (4.15), (4.16) and the fact that $(n-m)g(0)\neq 0$, it follows that

$$\sum_{i=1}^{n} M(p_i)=M(1)$$

valid for all $(p_1,\ldots,p_n)\in\Gamma_n$. By Result 1, there exists an additive mapping $A^*:\mathbb{R}\to\mathbb{R}$ such that $M(p)=A^*(p)$ for all $p\in I$ as $M(0)=0$. So, $M:I\to\mathbb{R}$ is additive on I. Now, from (4.13), (4.11), (4.16), the additivity of $b:\mathbb{R}\to\mathbb{R}$ and $M:I\to\mathbb{R}$ and the fact that $\sum_{t=1}^{n} g(r_t^*)-1\neq 0$, it follows that

$$1\neq\sum_{t=1}^{n} g(r_t^*)=M(1)-b(1)+ng(0)=1 \tag{4.17}$$

a contradiction. Hence our assumption is false. So, (4.4) holds. Now, (4.16) gives $M(1) = 1$. Also, from (4.14) and (4.4), now we get

$$\sum_{i=1}^{n}[M(p_i q) - M(q)M(p_i)] = 0.$$

Proceeding as in [16], we obtain $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$. In particular, $M(pq) = M(p)M(q)$ for all $p \in]0,1[$, $q \in]0,1[$. Thus $M : I \to \mathbb{R}$ is multiplicative. Since $M(0) = 0$, $M(1) = 1$, so $M : I \to \mathbb{R}$ is also a nonconstant mapping. Also, $M : I \to \mathbb{R}$ is nonadditive as if $M : I \to \mathbb{R}$ is additive, then again we get the contradiction (4.17). The required solution (4.3) now follows from (4.13) along with (4.11). □

Now we state the following:

Theorem 3. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $\bar{g} : I \to \mathbb{R}$ be a mapping which satisfies the functional equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\bar{g}(p_i q_j) = \sum_{i=1}^{n}\bar{g}(p_i)\sum_{j=1}^{m}\bar{g}(q_j) + n(m-1)\bar{g}(0) \tag{4.18}$$

for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$. Then $\bar{g}$ is of the form

$$\bar{g}(p) = \bar{a}(p) + \bar{g}(0) \tag{4.19}$$

where $\bar{a} : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$\bar{a}(1) = \begin{cases} -n\bar{g}(0) & \text{if } \bar{g}(1) + (m-1)\bar{g}(0) \neq 1 \\ 1 - m\bar{g}(0) & \text{if } \bar{g}(1) + (m-1)\bar{g}(0) = 1 \end{cases} \tag{4.20}$$

or

$$\bar{g}(p) = M(p) - \bar{b}(p) + \bar{g}(0) \tag{4.21}$$

with

$$(n-m)\bar{g}(0) = 0 \tag{4.22}$$

where $\bar{b} : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $\bar{b}(1) = n\bar{g}(0)$ and $M : I \to \mathbb{R}$ is a nonconstant nonadditive mapping which is multiplicative on I.

Now we prove the following:

Theorem 4. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $f : I \to \mathbb{R}$ be a mapping which satisfies the functional equation* (1.7)*, with $\lambda \neq 0$, for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$. Then f is of the form* (*for all $p \in I$*)

$$f(p) = \frac{1}{\lambda}[a(p) - p + \lambda f(0)]$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$a(1) = \begin{cases} -\lambda m f(0) & \text{if } f(1) + (n-1)f(0) \neq 0 \\ 1 - \lambda n f(0) & \text{if } f(1) + (n-1)f(0) = 0 \end{cases}$$

or

$$f(p) = \frac{1}{\lambda}[M(p) - p - b(p) + \lambda f(0)]$$

with

$$(n-m)f(0) = 0$$

where $b : \mathbb{R} \to \mathbb{R}$ *is an additive mapping with* $b(1) = \lambda m f(0)$ *and* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative on* I.

Proof. Define $g : I \to \mathbb{R}$ as $g(p) = \lambda f(p) + p$ for all $p \in I$. Then, (1.7) reduces to (1.6). Making use of Theorem 2, Theorem 4 follows. The details are omitted. □

5. The Functional Equations (1.8) and (1.9)

In this section, our object is to determine the general solutions of equations (1.8) and (1.9) valid for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers. To achieve this objective, we prove the following:

Lemma 5. *Let* $n \geq 3$, $m \geq 3$ *be fixed integers and* $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$ *be mappings which satisfy the functional equation (1.8) for all* $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. *If* $h(1) + (m-1)h(0) \neq 0$, *then the mapping* $\bar{g} : I \to \mathbb{R}$, *defined as*

$$\bar{g}(p) = [h(1) + (m-1)h(0)]^{-1} h(p) \tag{5.1}$$

for all $p \in I$, *satisfies the equation* (4.18) *with*

$$\bar{g}(1) + (m-1)\bar{g}(0) = 1. \tag{5.2}$$

On the other hand, if $h(1) + (n-1)h(0) \neq 0$, *then the mapping* $g : I \to \mathbb{R}$, *defined as*

$$g(p) = [h(1) + (n-1)h(0)]^{-1} h(p) \tag{5.3}$$

for all $p \in I$, *satisfies equation* (1.6) *in which* g *also satisfies* (4.6).

Proof. The case $h(1) + (m-1)h(0) \neq 0$. Let us put $q_1 = 1, q_2 = \ldots = q_m = 0$ in (1.8). We obtain

$$\sum_{i=1}^{n} \{f(p_i) - [h(1) + (m-1)h(0)]h(p_i)\} = -n(m-1)f(0).$$

By Result 1, there exists an additive mapping $\bar{A} : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = [h(1) + (m-1)h(0)][h(p) - h(0)] + \bar{A}(p) + f(0) \tag{5.4}$$

with

$$\bar{A}(1) = n\{[h(1) + (m-1)h(0)]h(0) - mf(0)\}. \tag{5.5}$$

From (1.8) and (5.4), we obtain

$$[h(1)+(m-1)h(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)-\sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}h(q_j) \tag{5.6}$$
$$= \quad n(m-1)h(0)[h(1)+(m-1)h(0)].$$

Define $\bar{g}: I \to \mathbb{R}$ as in (5.1). The elimination of h from (5.1) and (5.6) yields the equation (4.18). Also, from (5.1), it is easy to derive (5.2).

Now consider the case when $h(1)+(n-1)h(0) \neq 0$. In this case, we put $p_1 = 1, p_2 = \ldots = p_n = 0$ in (1.8) and obtain the equation

$$\sum_{j=1}^{m}\left\{f(q_j)-[h(1)+(n-1)h(0)]h(q_j)\right\} = -m(n-1)f(0).$$

By Result 1, there exists an additive mapping $A: \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = [h(1)+(n-1)h(0)][h(p)-h(0)]+A(p)+f(0) \tag{5.7}$$

with

$$A(1) = m\{[h(1)+(n-1)h(0)]h(0)-nf(0)\}. \tag{5.8}$$

From (1.8) and (5.7), we obtain

$$[h(1)+(n-1)h(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)-\sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}h(q_j) \tag{5.9}$$
$$= \quad m(n-1)h(0)[h(1)+(n-1)h(0)].$$

Define $g: I \to \mathbb{R}$ as in (5.3). The elimination of h from (5.3) and (5.9) yields equation (1.6). Also, (4.6) follows from (5.3). □

Now we prove the following:

Theorem 6. *Let* $n \geq 3$, $m \geq 3$ *be fixed integers and* $f: I \to \mathbb{R}$, $h: I \to \mathbb{R}$ *be mappings which satisfy the functional equation* (1.8) *for all* $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$. *Then any general solution* (f,h) *of* (1.8)*, for all* $p \in I$*, is of the form*

$$\begin{cases} f(p) = b_1(p)-\dfrac{1}{nm}b_1(1) \\ h(p) = b_2(p)-\dfrac{1}{n}b_2(1) \end{cases} \tag{5.10}$$

or

$$\begin{cases} f(p) = b_1(p)-\dfrac{1}{nm}b_1(1) \\ h(p) = b_2(p)-\dfrac{1}{m}b_2(1) \end{cases} \tag{5.11}$$

or

$$\begin{cases} f(p) = [h(1) + (m-1)h(0)]^2 \bar{a}(p) + \bar{A}(p) + f(0) \\ h(p) = [h(1) + (m-1)h(0)]\bar{a}(p) + h(0) \end{cases} \tag{5.12}$$

or

$$\begin{cases} f(p) = [h(1) + (n-1)h(0)]^2 a(p) + A(p) + f(0) \\ h(p) = [h(1) + (n-1)h(0)]a(p) + h(0) \end{cases} \tag{5.13}$$

or

$$\begin{cases} f(p) = [h(1) + (m-1)h(0)]^2 [M(p) - \bar{b}(p)] + \bar{A}(p) + f(0) \\ h(p) = [h(1) + (m-1)h(0)][M(p) - \bar{b}(p)] + h(0) \end{cases} \tag{5.14}$$

or

$$\begin{cases} f(p) = [h(1) + (n-1)h(0)]^2 [M(p) - b(p)] + A(p) + f(0) \\ h(p) = [h(1) + (n-1)h(0)][M(p) - b(p)] + h(0) \end{cases} \tag{5.15}$$

with

$$(n-m)h(0) = 0 \tag{5.16}$$

in (5.14) *and* (5.15)*;* $M : I \to \mathbb{R}$ *is a nonconstant nonadditive mapping which is multiplicative on* $I; A : \mathbb{R} \to \mathbb{R}$, $\bar{A} : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\bar{b} : \mathbb{R} \to \mathbb{R}$*;* $b_i : \mathbb{R} \to \mathbb{R} (i = 1, 2)$, $a : \mathbb{R} \to \mathbb{R}$, $\bar{a} : \mathbb{R} \to \mathbb{R}$ *are additive mappings such that* $\bar{A}(1)$ *and* $A(1)$ *are given by* (5.5) *and* (5.8) *and, moreover,*

$$\begin{cases} \bar{a}(1) = 1 - mh(0)[h(1) + (m-1)h(0)]^{-1} \\ a(1) = 1 - nh(0)[h(1) + (n-1)h(0)]^{-1} \\ \bar{b}(1) = nh(0)[h(1) + (m-1)h(0)]^{-1} \\ b(1) = mh(0)[h(1) + (n-1)h(0)]^{-1}, \end{cases} \tag{5.17}$$

$f(0)$ *and* $h(0)$ *being arbitrary constants and also* $h(1) + (m-1)h(0) \neq 0$, $h(1) + (n-1)h(0) \neq 0$ *in* (5.17) *and* (5.12) *to* (5.15).

Proof. We divide our discussion into three cases:

Case 1. $\sum_{i=1}^{n} h(p_i) = 0$ for all $(p_1, \ldots, p_n) \in \Gamma_n$.

In this case, by Result 1, $h(p) = b_2(p) - \frac{1}{n} b_2(1)$ where $b_2 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Also, (1.8) reduces to $\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = 0$ whose solution is $f(p) = b_1(p) - \frac{1}{nm} b_1(1)$, where $b_1 : \mathbb{R} \to \mathbb{R}$ is an additive mapping. Thus, we have obtained the solution (5.10) of (1.8).

Case 2. $\sum_{j=1}^{m} h(q_j) = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$.

In this case, we obtain the solution (5.11). The proof is similar.

Case 3. Neither $\sum_{i=1}^{n} h(p_i)$ vanishes identically on Γ_n nor $\sum_{j=1}^{m} h(q_j)$ vanishes identically on Γ_m. Then there exists a probability distribution $(p_1^*,\ldots,p_n^*) \in \Gamma_n$ and a $(q_1^*,\ldots,q_m^*) \in \Gamma_m$ such that $\sum_{i=1}^{n} h(p_i^*) \neq 0$ and $\sum_{j=1}^{m} h(q_j^*) \neq 0$. Then $\sum_{i=1}^{n} h(p_i^*) \sum_{j=1}^{m} h(q_j^*) \neq 0$. Now we prove that $h(1)+(m-1)h(0) \neq 0$. If possible, suppose $h(1)+(m-1)h(0)=0$. Then (5.6) reduces to $\sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} h(q_j) = 0$ for all $(p_1,\ldots,p_n) \in \Gamma_n$ and $(q_1,\ldots,q_m) \in \Gamma_m$ there by contradicting $\sum_{i=1}^{n} h(p_i^*) \sum_{j=1}^{m} h(q_j^*) \neq 0$. Similarly, we can prove that $h(1)+(n-1)h(0) \neq 0$. The solutions (5.12) to (5.15); along with (5.5), (5.8), (5.16) and (5.17); follow from Lemma 5, Theorems 2 and 3. □

Now we mention the following:

Theorem 7. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $F: I \to \mathbb{R}$, $H: I \to \mathbb{R}$ be mappings which satisfy the functional equation* (1.9)*, with $\lambda \neq 0$, for all $(p_1,\ldots,p_n) \in \Gamma_n$, $(q_1,\ldots,q_m) \in \Gamma_m$. Then, any general solution (F,H) of* (1.9)*, for all $p \in I$ is of the form*

$$\begin{cases} F(p) = \dfrac{1}{\lambda}[b_1(p) - p - \dfrac{1}{nm} b_1(1)] \\ H(p) = \dfrac{1}{\lambda}[b_2(p) - p - \dfrac{1}{n} b_2(1)] \end{cases} \tag{5.18}$$

or

$$\begin{cases} F(p) = \dfrac{1}{\lambda}[b_1(p) - p - \dfrac{1}{nm} b_1(1)] \\ H(p) = \dfrac{1}{\lambda}[b_2(p) - p - \dfrac{1}{m} b_2(1)] \end{cases} \tag{5.19}$$

or

$$\begin{cases} F(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(m-1)H(0)]\}^2 \bar{a}(p) + \bar{A}(p) - p + \lambda F(0)\right) \\ H(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(m-1)H(0)]\}\bar{a}(p) - p + \lambda H(0)\right) \end{cases} \tag{5.20}$$

or

$$\begin{cases} F(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(n-1)H(0)]\}^2 a(p) + A(p) - p + \lambda F(0)\right) \\ H(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(n-1)H(0)]\}a(p) - p + \lambda H(0)\right) \end{cases} \tag{5.21}$$

or

$$\begin{cases} F(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(m-1)H(0)]\}^2 [M(p) - \bar{b}(p)] + \bar{A}(p) - p + \lambda F(0)\right) \\ H(p) = \dfrac{1}{\lambda}\left(\{1+\lambda[H(1)+(m-1)H(0)]\}[M(p) - \bar{b}(p)] - p + \lambda H(0)\right) \end{cases} \tag{5.22}$$

or

$$\begin{cases} F(p) = \frac{1}{\lambda}\left(\{1+\lambda[H(1)+(n-1)H(0)]\}^2[M(p)-b(p)]+A(p)-p+\lambda F(0)\right) \\ H(p) = \frac{1}{\lambda}\left(\{1+\lambda[H(1)+(n-1)H(0)]\}[M(p)-b(p)]-p+\lambda H(0)\right) \end{cases} \tag{5.23}$$

with

$$\begin{cases} (n-m)H(0) = 0 \\ 1+\lambda[H(1)+(m-1)H(0)] \neq 0, \quad 1+\lambda[H(1)+(n-1)H(0)] \neq 0, \end{cases} \tag{5.24}$$

$A(1)$, $\bar{A}(1)$, $a(1)$, $\bar{a}(1)$, $b(1)$, $\bar{b}(1)$ *given by* (5.8), (5.5) *and* (5.17), *the mapping* $M : I \to \mathbb{R}$ *being nonconstant nonadditive and multiplicative; the mappings* A, $\bar{A}$, a, $\bar{a}$, b, $\bar{b}$ *and* b_i $(i = 1,2)$ *being additive on* $\mathbb{R}$.

Proof. Define $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ as

$$f(p) = p + \lambda F(p), \qquad h(p) = p + \lambda H(p) \tag{5.25}$$

for all $p \in I$ and $\lambda \neq 0$. Then

$$F(p) = \frac{1}{\lambda}[f(p)-p], \qquad H(p) = \frac{1}{\lambda}[h(p)-p]. \tag{5.26}$$

Theorem 7 now follows from Theorem 6 and equations (5.26). The details are omitted. □

Remarks. 1. Theorems 2, 6 and 7 are, indeed, generalizations of Theorems 1, 2 and 3 proved by the authors [15] using the notations k and l in place of n and m. Theorem 2 differs considerably from Theorem 1 proved in [15]. Indeed, in [15], we had $M(1) = g(1)+(m-1)g(0)$. Thus, it seems that $M(1)$ depends upon $g(0)$, $g(1)$ and the fixed integer $m \geq 3$. Since $M(1) = g(1)+(m-1)g(0)$ and $g(1)+(n-1)g(0) = 1$, it follows that $M(1) = 1-(n-m)g(0)$. In this paper, we have proved that indeed, $M(1) = 1$ and $(n-m)g(0) = 0$. Consequently, Theorems 2 and 3 proved in [15] have been improved accordingly.

2. The functional equation (2.4) reduces to (1.6) if the second term appearing on its right hand side is omitted.

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