A Brief Introduction to Greedy Algorithms

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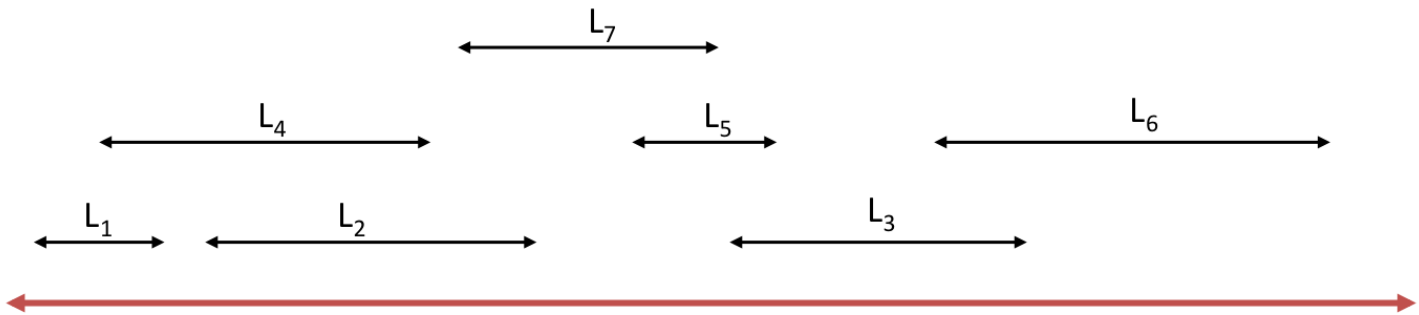
Greedy Algorithms

- "*A local (greedy) decision rule leads to a globally optimal solution.*"

Problem

Interval scheduling: Given a set of n intervals of the form $(S(i), F(i))$, find the largest subset of non-overlapping intervals.



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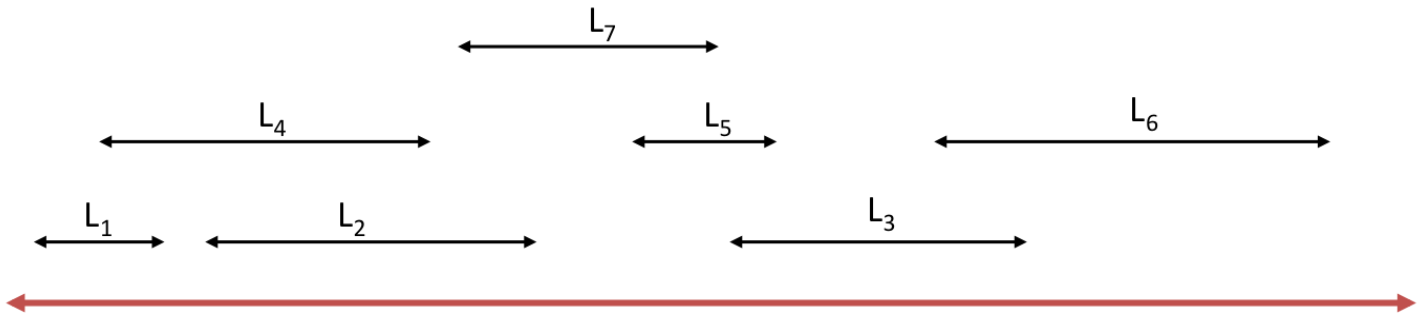
Interval scheduling: Given a set of n intervals of the form $(S(i), F(i))$, find the largest subset of non-overlapping intervals.

- Candidate greedy choices:
  - Earliest start time
  - Smallest duration
  - Least overlapping



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- Candidate greedy choices:
  - Earliest start time
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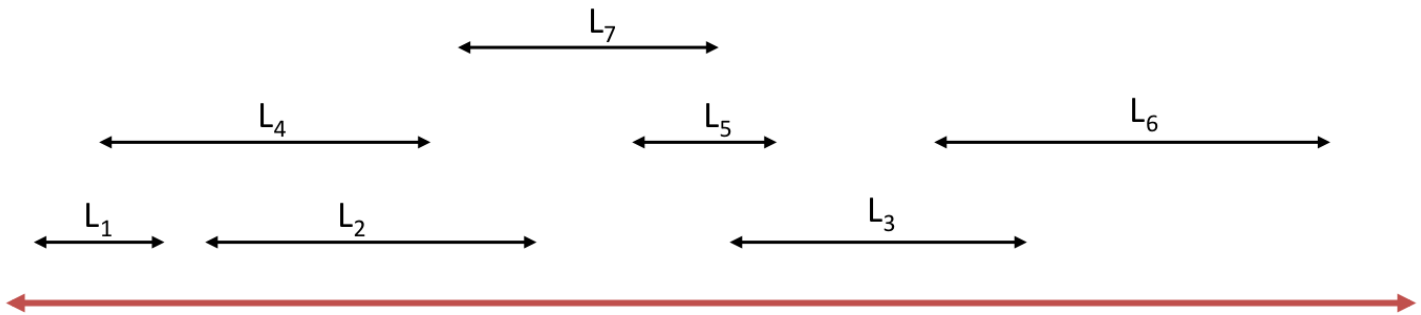
  - Least overlapping
  - Earliest finish time



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Algorithm

`GreedySchedule`

- Initialize R to contain all intervals
- While R is not empty
  - Choose an interval $(S(i), F(i))$ from R that has the smallest value of $F(i)$
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- Yes we can! We will use "greedy stays ahead" method to show this.

Proof

Let $a_1, a_2, ..., a_k$ be the sequence of requests that `GreedySchedule` picks and $o_1, o_2, ..., o_l$ be the requests in O sorted in non-decreasing order by finishing time.

- Claim 1: $F(a_1) \leq F(o_1)$.

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- Claim 1: $F(a_1) \leq F(o_1)$.
- Claim 2: If $F(a_1) \leq F(o_1)$, $F(a_2) \leq F(o_2)$, ..., $F(a_{i-1}) \leq F(o_{i-1})$, then $F(a_i) \leq F(o_i)$.

- Question: Let O denote some optimal subset and A by the subset given by `GreedySchedule`. Can we show that $A = O$?
- Question Can we show that $|O| = |A|$?
- Yes we can! We will use "greedy stays ahead" method to show this.

Proof

- Let $a_1, a_2, ..., a_k$ be the sequence of requests that `GreedySchedule` picks and $o_1, o_2, ..., o_l$ be the requests in O sorted in non-decreasing order by finishing time.
- We will show by induction that $\forall i, F(a_i) \leq F(o_i)$
- Claim 1 (base case): $F(a_1) \leq F(o_1)$.
- Claim 2 (inductive step): If $F(a_1) \leq F(o_1)$, $F(a_2) \leq F(o_2)$, ..., $F(a_{i-1}) \leq F(o_{i-1})$, then $F(a_i) \leq F(o_i)$.
- `GreedySchedule` could not have stopped after a_k.

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- Running time?

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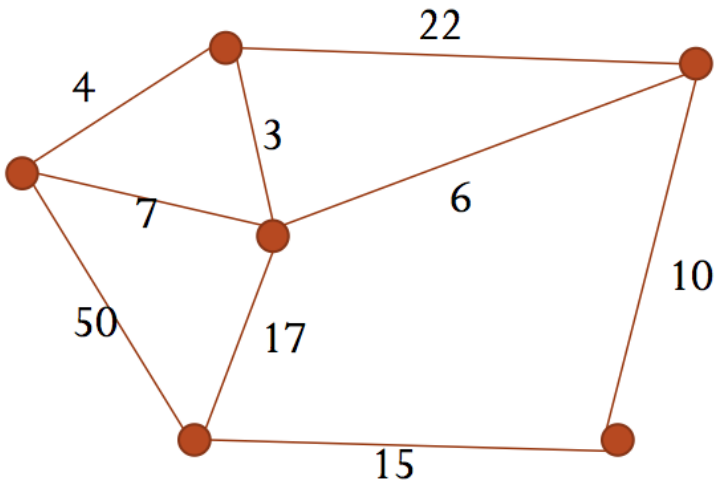
- Running time? $O(n \log n)$

- Spanning Tree: Given a strongly connected graph $G = (V, E)$, a *spanning tree* of G is a subgraph $G' = (V, E')$ such that G' is a tree.
- Minimum Spanning Tree (MST): Given a strongly connected weighted graph $G = (V, E)$, a *Minimum Spanning Tree* of G is a spanning tree of G of minimum total weight (i.e., sum of weight of edges in the tree).



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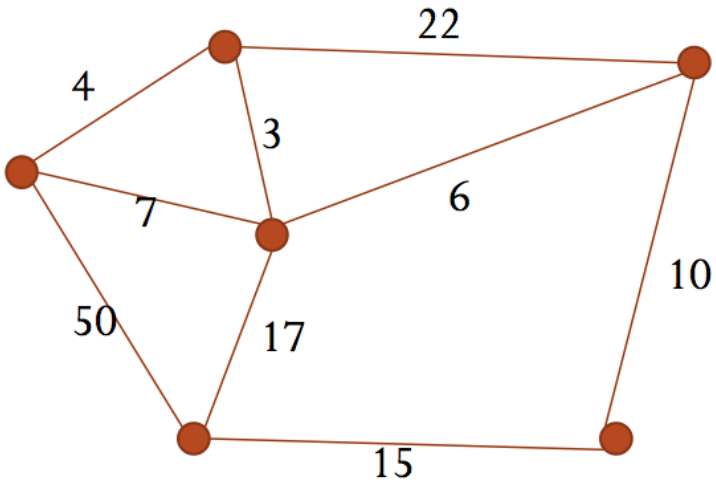
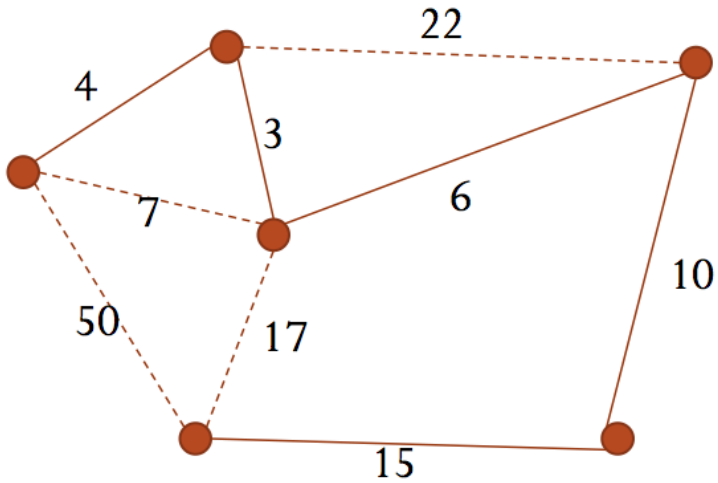
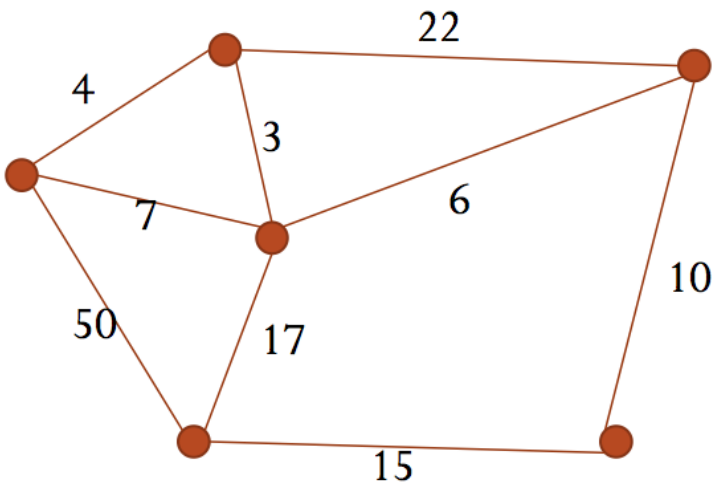
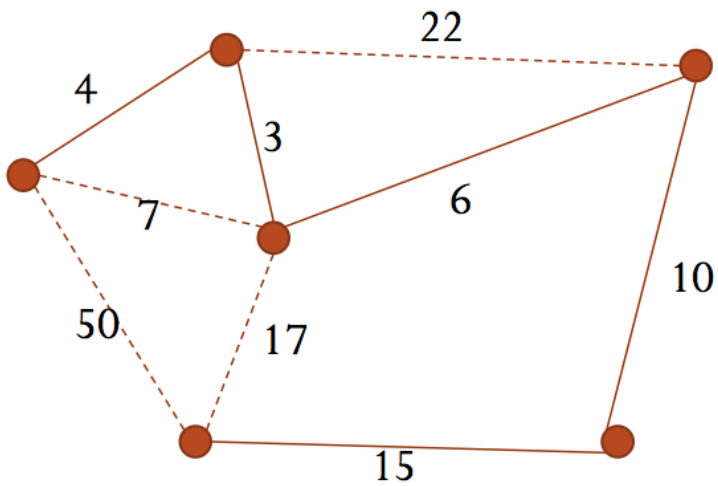




Problem

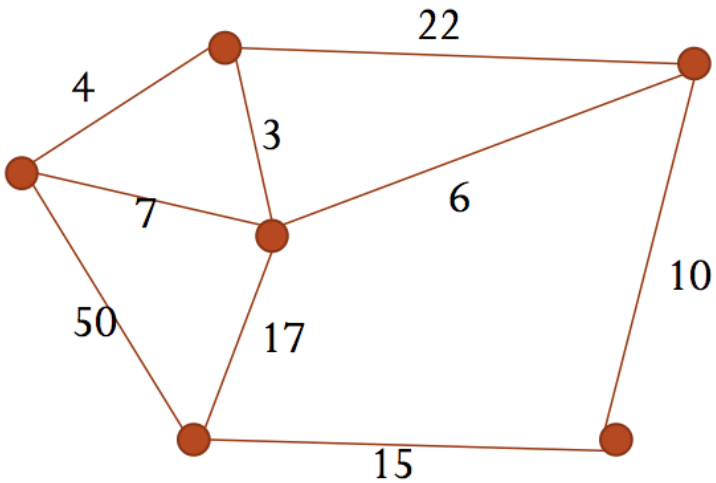
Given a weighted graph G where all the edge weights are distinct, give an algorithm for finding the MST of G.

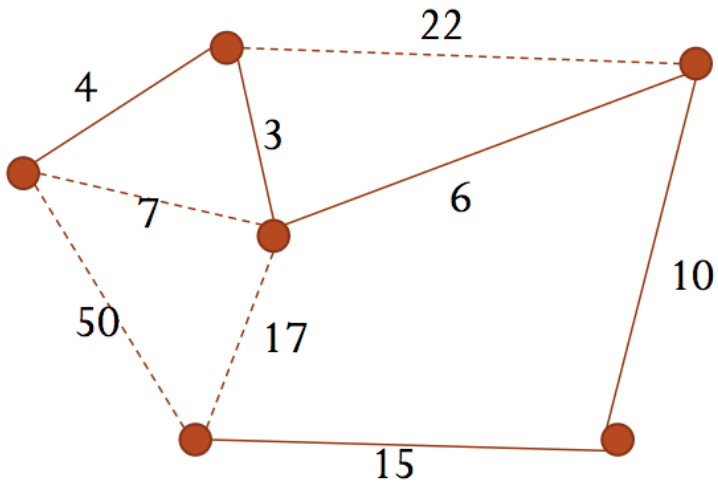




Theorem

Cut property: Given a weighted graph $G = (V, E)$ where all the edge weights are distinct. Consider a non-empty proper subset S of V and $S' = V \setminus S$. Let e be the least weighted edge between any pair of vertices (u, v), where u is in S and v is in S'. Then e is necessarily present in *all* MSTs of G.





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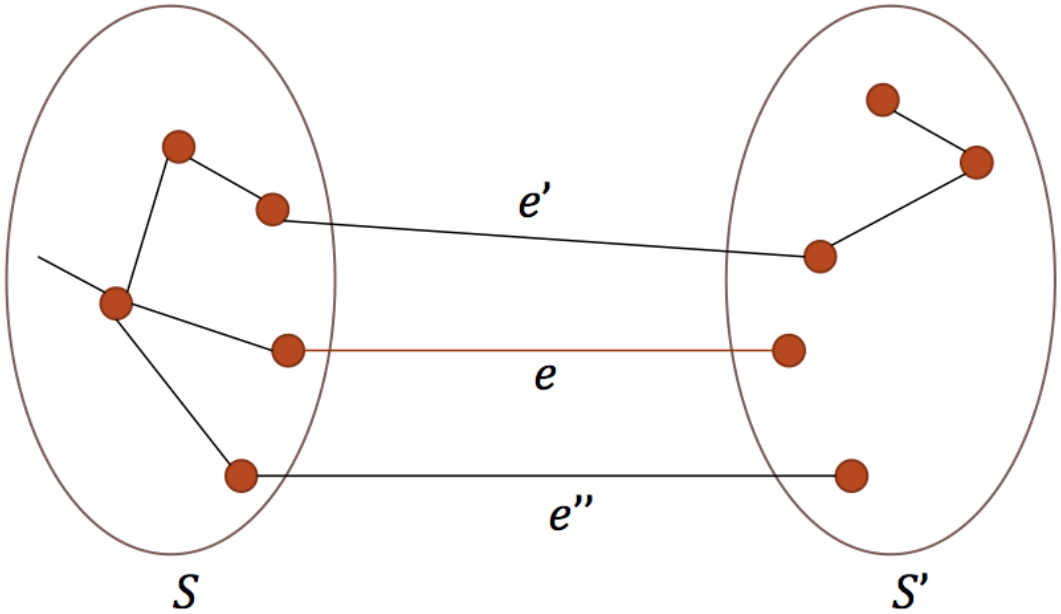

Algorithm

`Prim's Algorithm(G)`

- $S \leftarrow \{u\}$ //*u is an arbitrary vertex in the graph*
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  - Let $e = (v, w)$ be the minimum weight edge between S and $V \setminus S$
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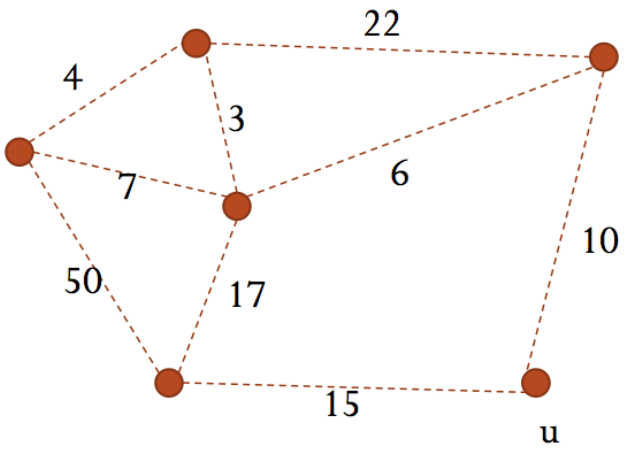

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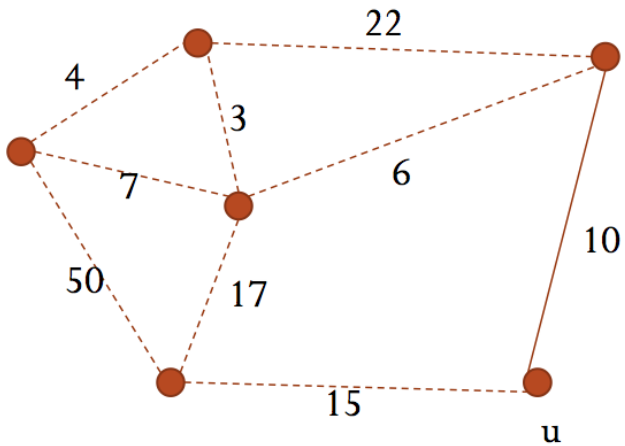

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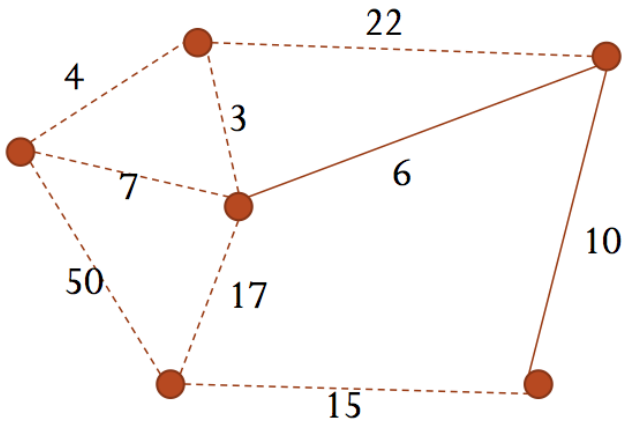

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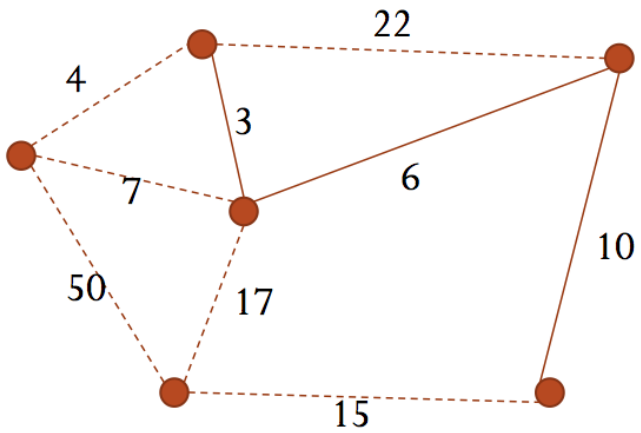

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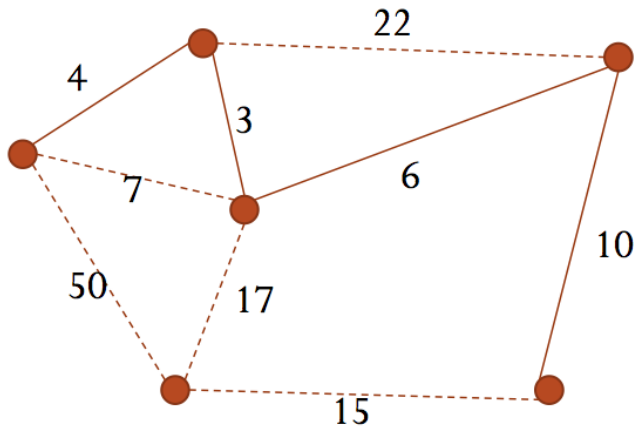

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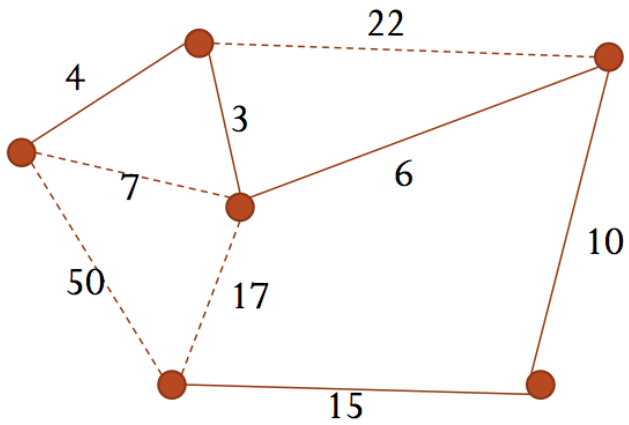

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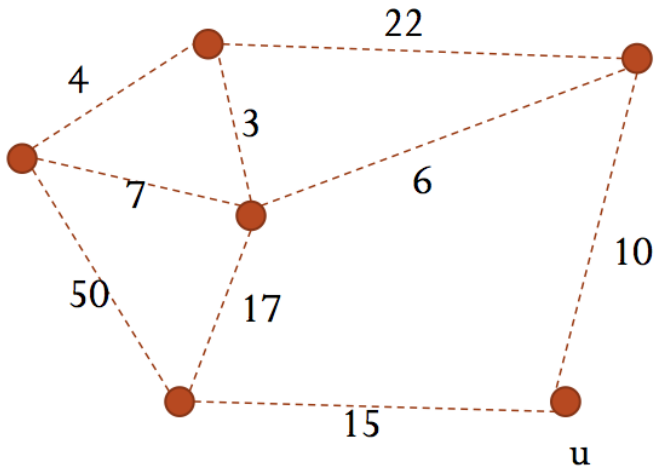

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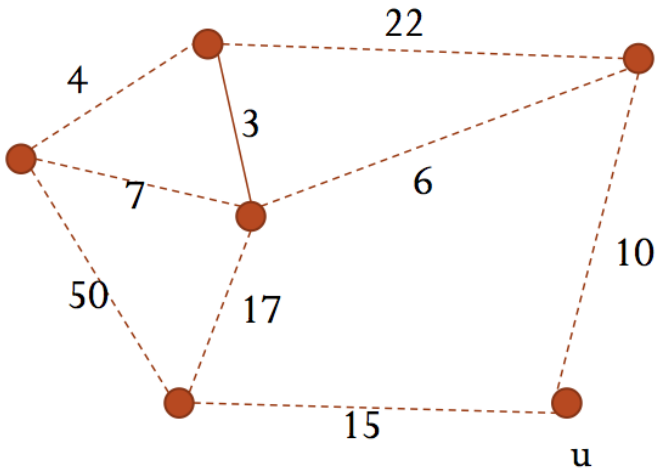

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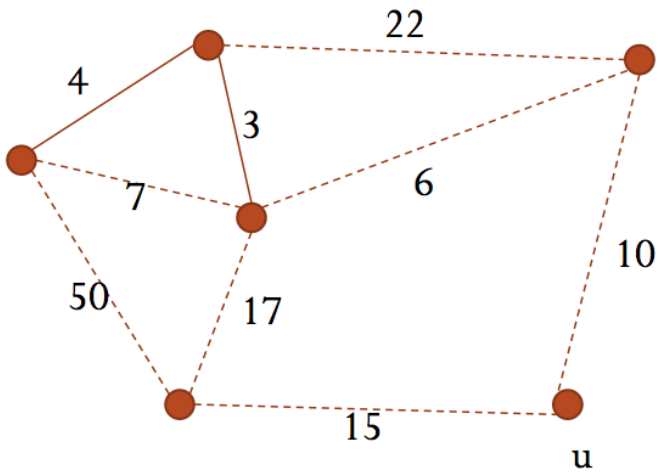

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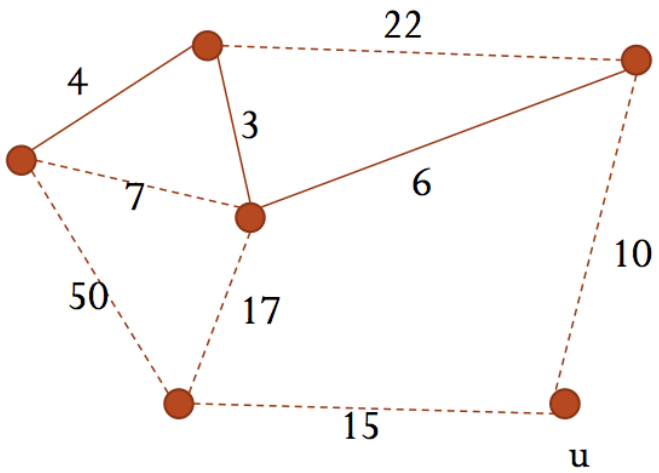

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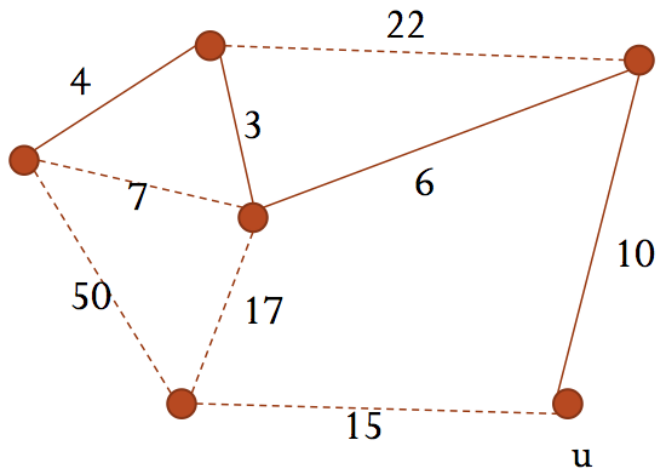

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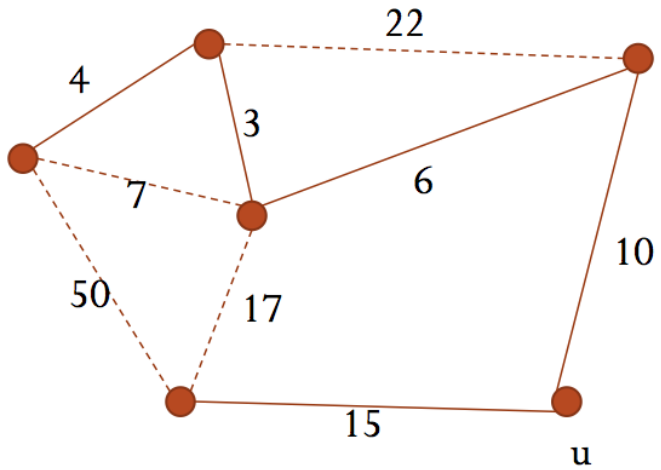

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- What is the running time of Prim's algorithm?

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- $S \leftarrow E$; $T \leftarrow \{\}$
- While the edge set T does not connect all the vertices
  - *//Note that $G' = (V, T)$ contains dicsonnected components*
  - Let e be the minimum weight edge in the set S
  - ~~If e does not create a cycle in T~~
  - If u and v are in different components of G'
    - $T \leftarrow T \cup \{e\}$
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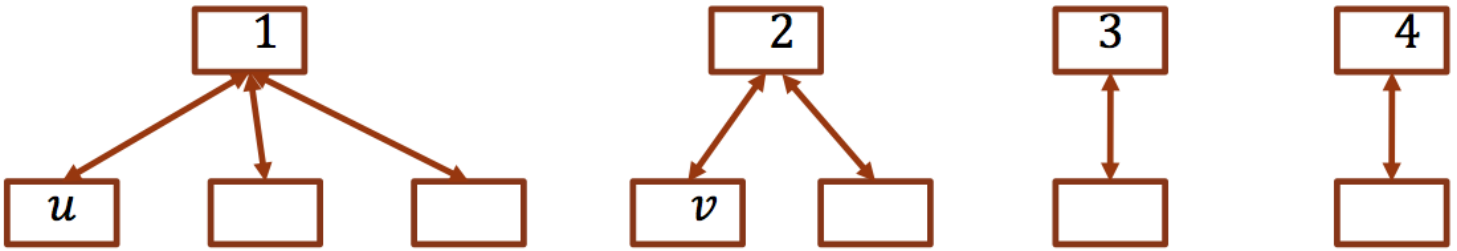
- Union-Find: Used for storing partition of a set of elements. The following two operations are supported:
  1. *Find*(v): Find the partition to which the element v belongs.
  2. *Union*(u, v): Merge the partition to which u belongs with the partition to which v belongs.
- Consider the following data structure.



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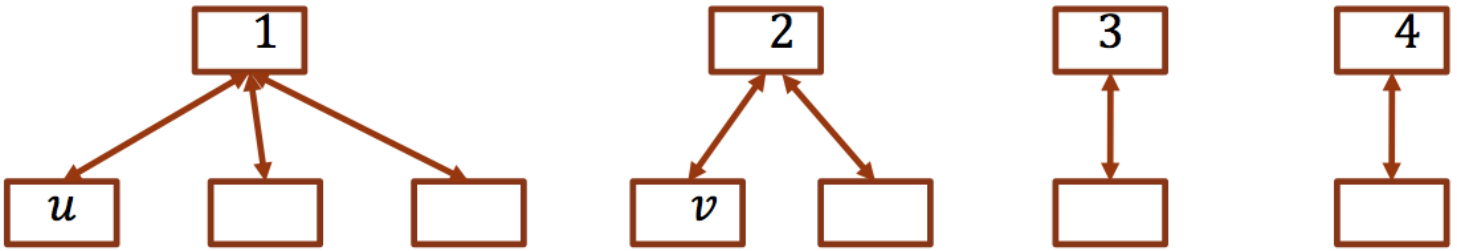

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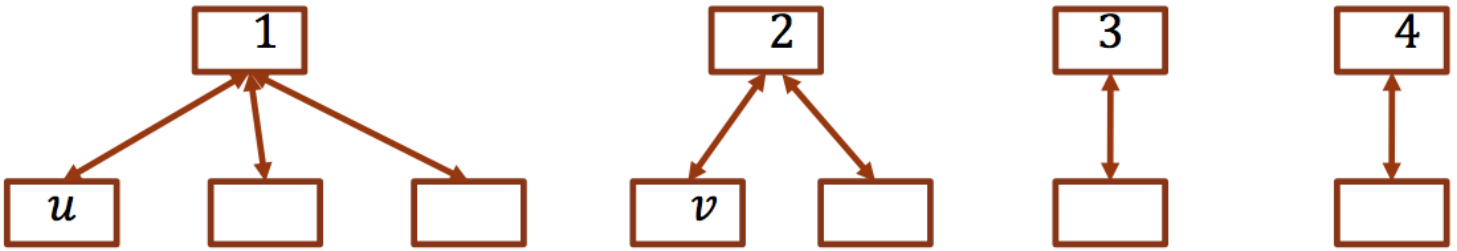

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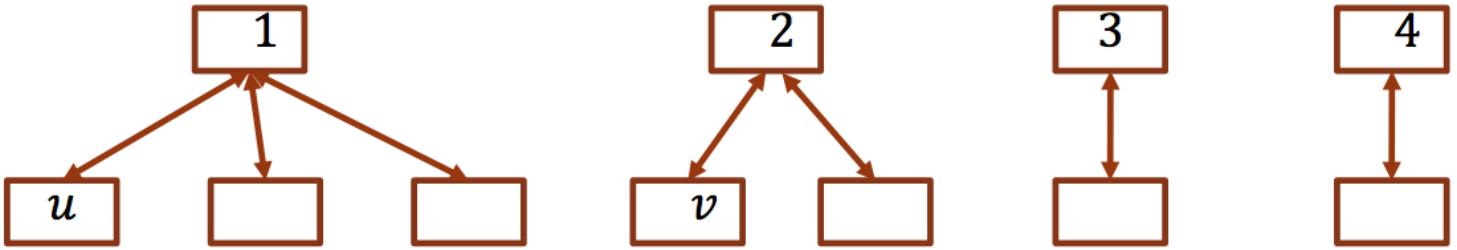
    - Claim: Performing k union operations takes $O(k \log k)$ time in the worst case.



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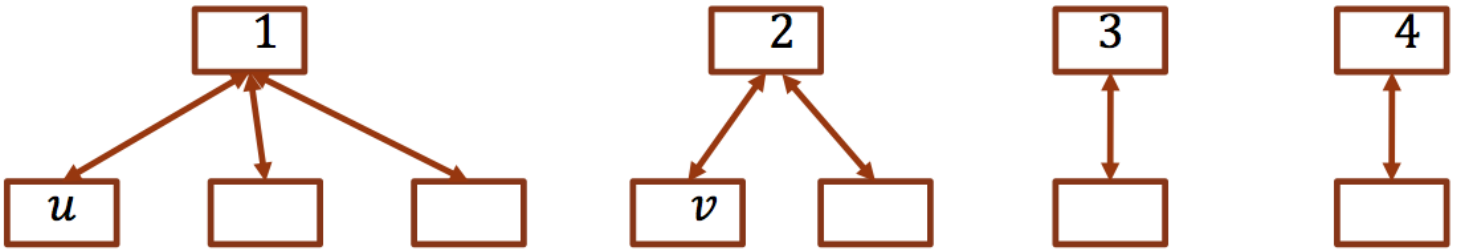

- Suppose we start from a full partition (i.e., each partition contains one element).
- How much time does the following operation take:
  - *Find*(v):
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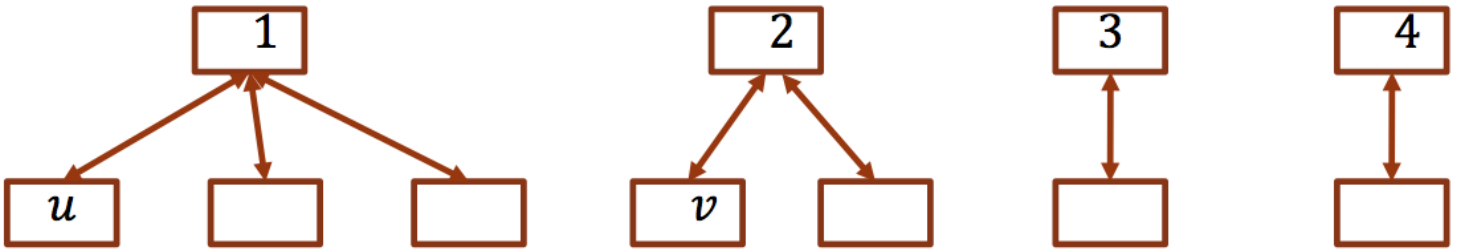

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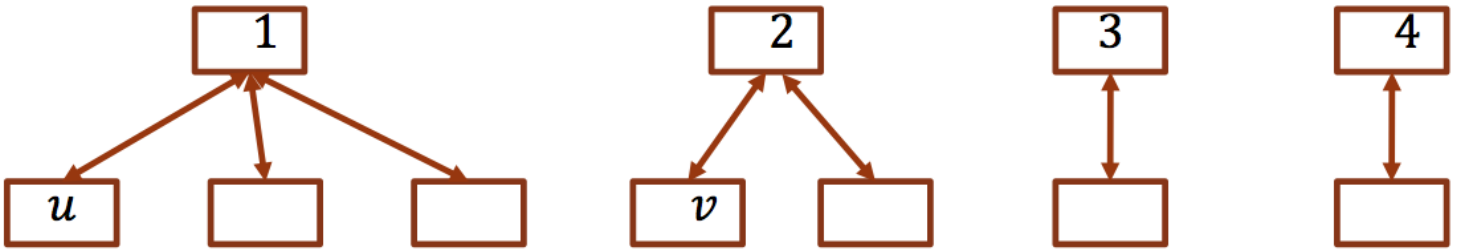

- Suppose we start from a full partition (i.e., each partition contains one element).
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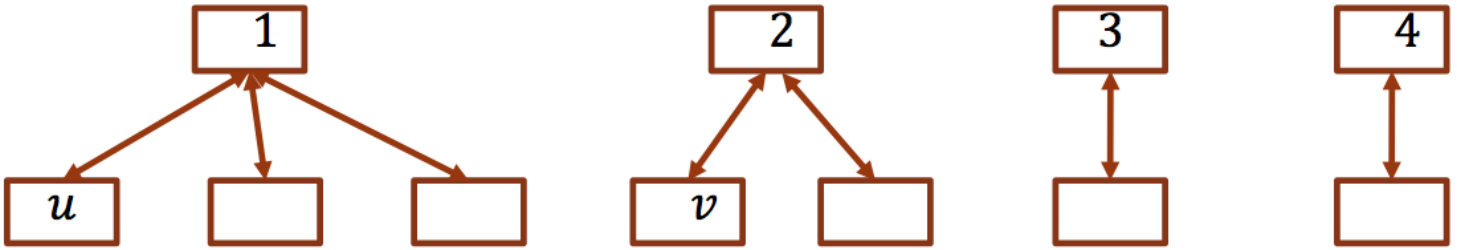
    - Claim: Performing k union operations takes $O(k \log k)$ time in the worst case.



- Kruskal's algorithm using Union-Find.

Algorithm

```
Kruskal's Algorithm(G)
```
- $S \leftarrow E$; $T \leftarrow \{\}$
- While the edge set T does not connect all the vertices
  - *//Note that $G' = (V, T)$ contains dicsonnected components*
  - Let e be the minimum weight edge in the set S
  - ~~If e does not create a cycle in T~~
  - ~~If u and v are in different components of G'~~
  - If $(Find(u) \neq Find(v))$
    - $T \leftarrow T \cup \{e\}$
    - $Union(u, v)$
  - $S \leftarrow S \setminus \{e\}$

- What is the running time of the above algorithm?

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- What is the running time of the above algorithm? $O(|E| \cdot \log |V|)$

- Path length: Let $G = (V, E)$ be a weighted directed graph. Given a path in G, the length of a path is defined to be the sum of lengths of the edges in the path.
- Shortest path: The shortest path from u to v is the path with minimum length.

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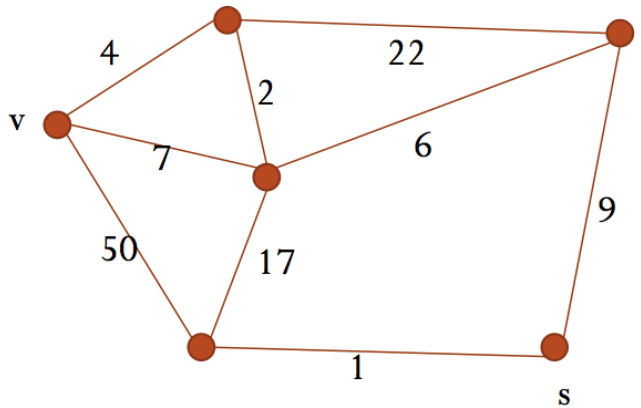
Single source shortest path: Given a weighted, directed graph $G = (V, E)$ with positive edge weights and a source vertex s, find the shortest path from s to all other vertices in the graph.



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- Claim 1: Shortest path is a *simple* path.
- Claim 2: Let S be a subset of vertices containing s such that we know the shortest path length $l(s, u)$ from s to any vertex in $u \in S$. Let $e = (u, v)$ be an edge such that
  1. $u \in S$, $v \in V \setminus S$,
  2. $(l(s, u) + W_e)$ is the least among all such cut edges.

  Then $l(s, v) = l(s, u) + W_e$.

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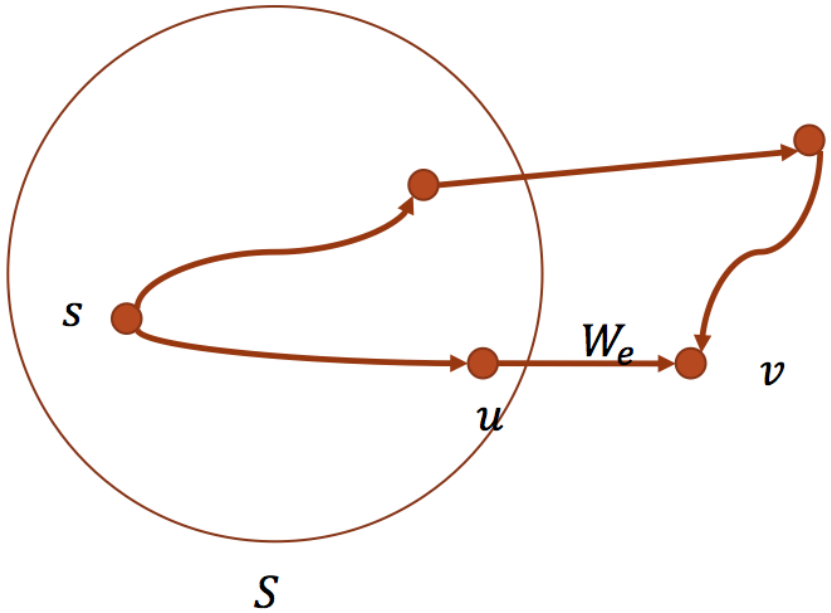

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Algorithm

`Dijkstra's Algorithm(`G, s`)`

- $S \leftarrow \{s\}$
- $d(s) \leftarrow 0$
- While S does not contain all vertices in G
  - Let $e = (u,v)$ be a cut edge across $(S, V \setminus S)$ with minimum value of $d(u) + W_e$
  - $d(v) \leftarrow d(u) + W_e$
  - $S \leftarrow S \cup \{v\}$

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- What is the running time of the above algorithm?

Greedy (Approximation) Algorithms

Greedy (Approximation) Algorithms

- For some problems, even though the greedy strategy does not give an optimal solution but it might give a solution that is provably close to the optimal solution.

- Covering set: Let S be a set containing n elements. A set of subsets $\{S_1, ..., S_m\}$ of S is called a covering set if each element in S is present in at least one of the subsets $S_1, ..., S_m$.

Problem

Set Cover: Given a set S containing n elements and m subsets $S_1, ..., S_m$ of S. Find a covering set of S of minimum cardinality.

- Example
  - $S = \{a, b, c, d, e, f\}$
  - $S_1 = \{a, b\}$, $S_2 = \{a, c\}$, $S_3 = \{b, c\}$, $S_4 = \{d, e, f\}$, $S_5 = \{e, f\}$
  - $\{S_1, S_2, S_3, S_4\}$ is a covering set.
  - $\{S_1, S_2, S_4\}$ is a covering set of minimum cardinality.

Problem

Set Cover: Given a set S containing n elements and m subsets $S_1, ..., S_m$ of S. Find a covering set of S of minimum cardinality.

- Application: There are n villages and the government is trying to figure out which villages to open schools at so that it has to open minimum number of schools. The constraint is that no children should have to walk more than 3 miles to get to a school.

Problem

Set Cover: Given a set S containing n elements and m subsets $S_1, ..., S_m$ of S. Find a covering set of S of minimum cardinality.

- Greedy strategy: Give preference to the subsets that covers the most number of (remaining) elements.

Algorithm

`GreedySetCover`$(S, S_1, ..., S_m)$

- $T \leftarrow \{\}$; $R \leftarrow S$
- While R is not empty:
  - Pick a subset S_i that covers the maximum number of elements in R
  - $T \leftarrow T \cup \{S_i\}$; $R \leftarrow R - S_i$

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- Counterexample: $S = \{a, b, c, d, e, f, g, h\}, S_1 = \{a, b, c, d, e\}, S_2 = \{a, b, c, f\}, S_3 = \{d, e, g, h\}$.

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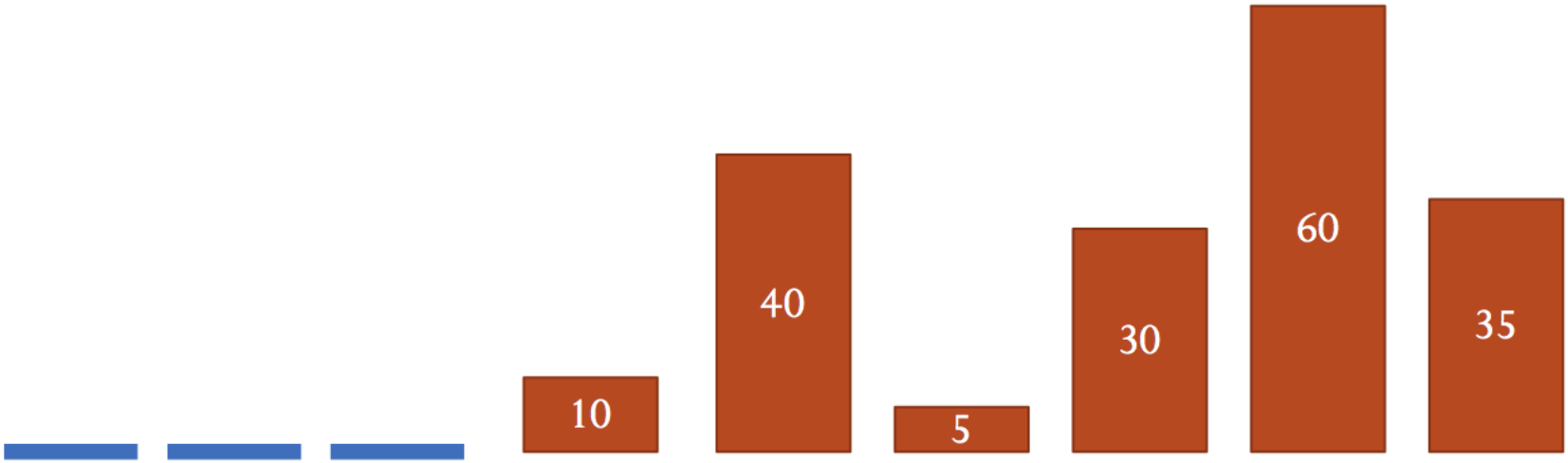
- Claim 1.2: $N_{k \cdot \ln n} < 1$.
  - Use the fact that $(1 - x) \leq e^{-x}$ and the equality holds only for $x = 0$.

Problem

Minimum Makespan: You have m identical machines and n jobs. For each job i, you are given the duration of this job $d(i)$ that denotes the time that is required by any machine to perform this job. Assign these n jobs on m machine such that the maximum finishing time is minimized.



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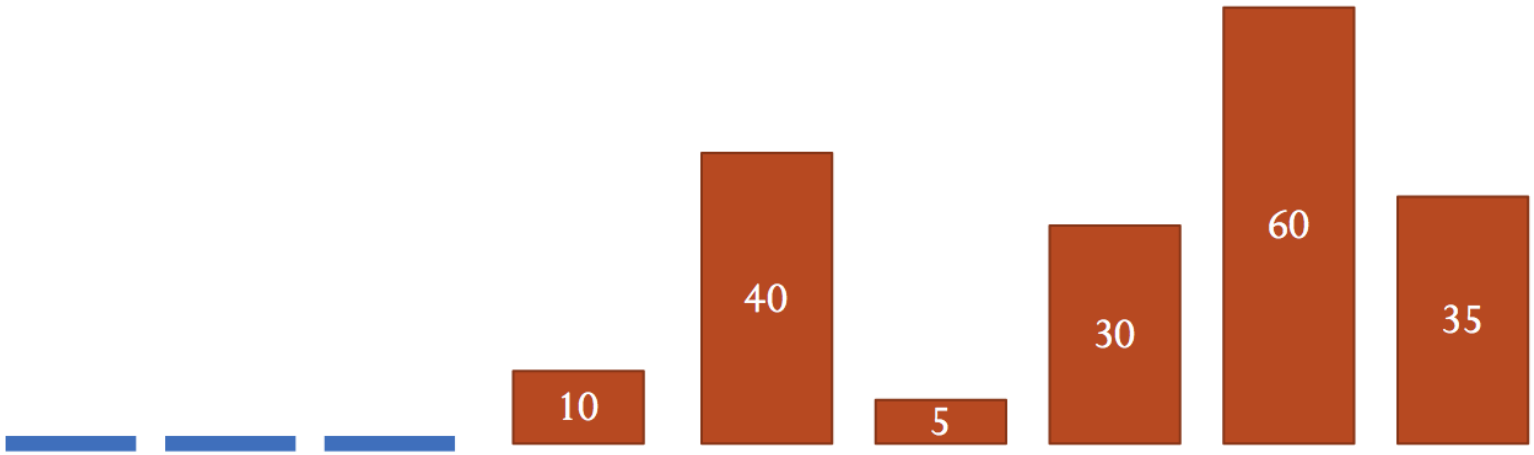
- Greedy strategy: Assign the next job to a machine with least load.



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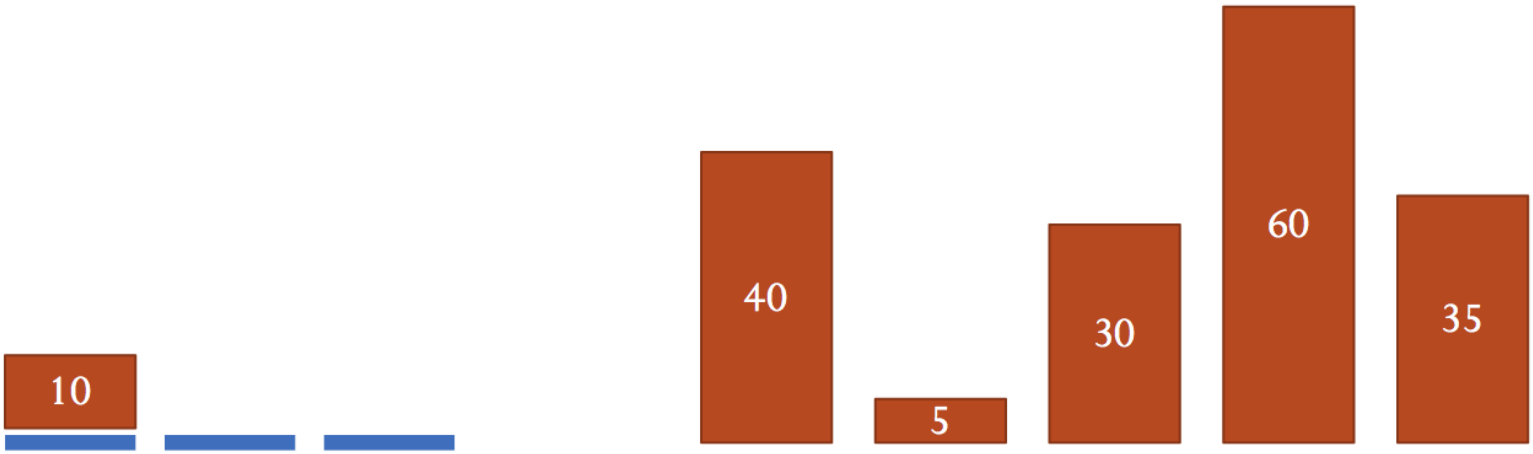

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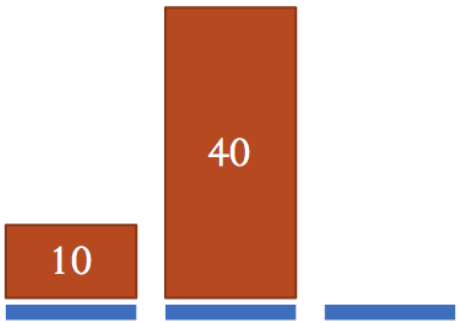

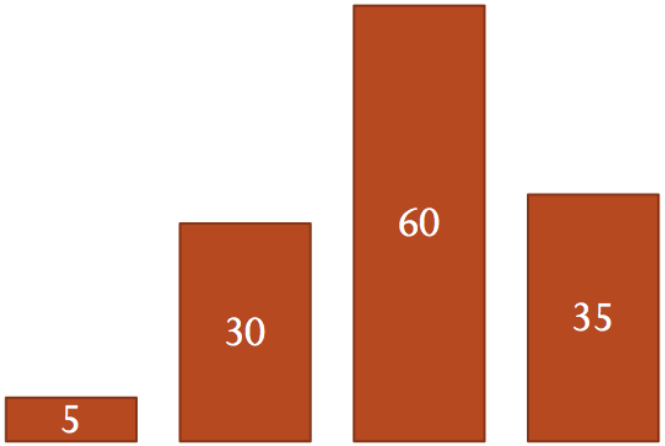

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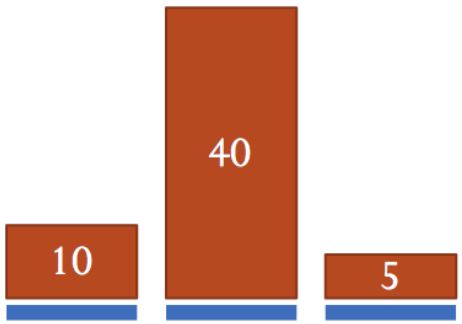

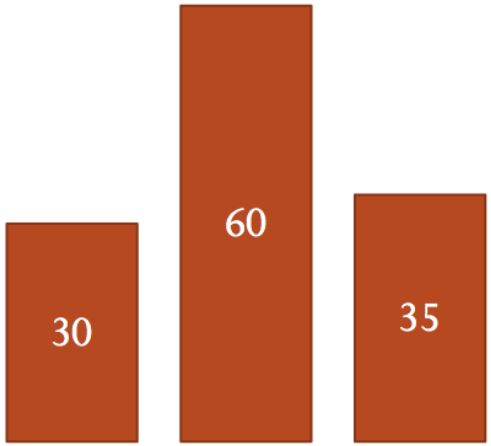

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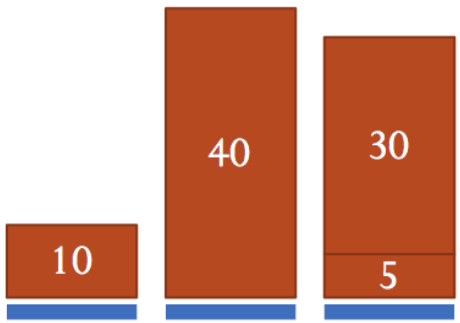

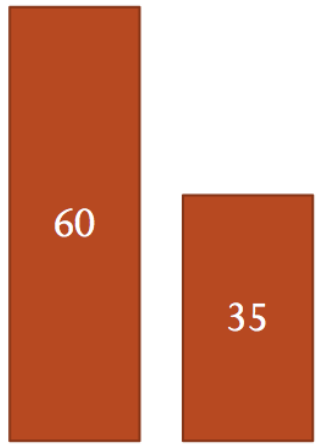

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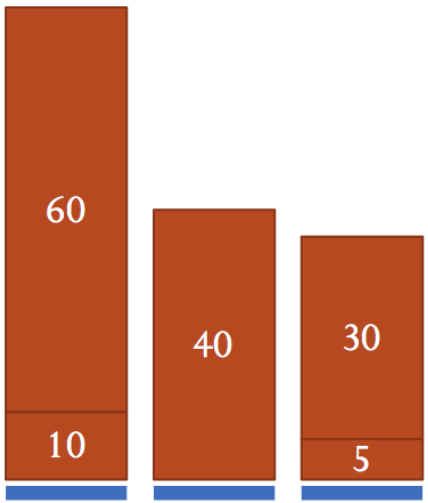

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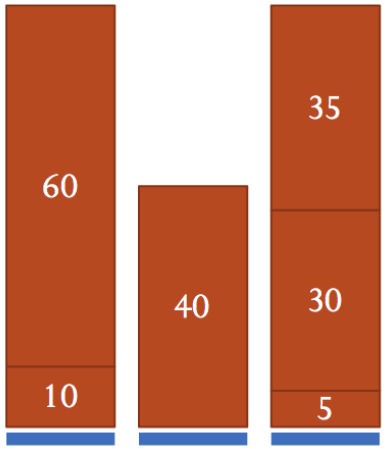

- Is this solution optimal?

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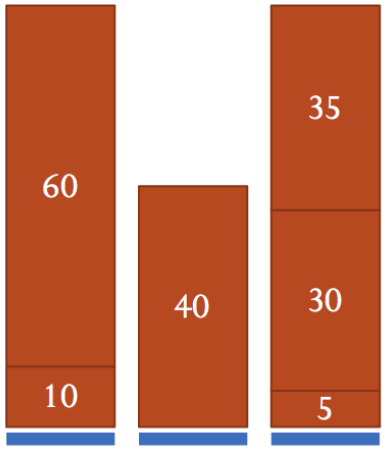

- Is this solution optimal? No

Algorithm

`GreedyMakespan`

- While all jobs are not assigned
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- Let OPT be the optimal value.
- Let G denote the maximum finishing time of a machine as per the greedy assignment.
- Claim 1: $G \leq 2 \cdot OPT$.

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End