# GENERATION AND CHARACTERISATION OF STATISTICAL FRACTALS

Statistics and Computer Application-PHYS-601

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**Abstract**

Using numerical methods, here I have generated Levy Flight Distribution (which are Statistical Fractals) using both Analytic and Fourier methods. The compatibility and accuracy of these methods are shown and discussed. We have used the values $\alpha = 0.6$ and $\alpha = 1.2$ for drawing a comparison of these methods. For the calculation of the fractal dimension, I have used The General Method and evaluated the same for $\alpha = 1.20$, with 18.33 %relative error which shows Analytic Method to be fair enough for Generating Levy Flight distribution.

# Contents

# Chapter 1

# INTRODUCTION

Fractals are found everywhere in nature. From the similar leaf patters to large scale stucture like galaxies and galaxy clusters, fractals show their existence every where. A fractal is a mathematical set that has a fractal dimension which characterises it and which can be fractional. Fractals are self-similar patterns, where self-similarity

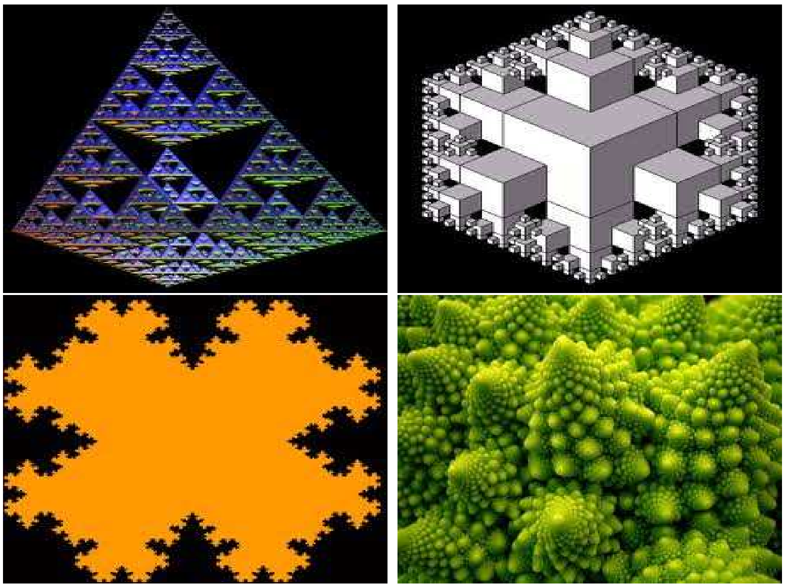

Figure 1.1: Different types of fractals

extends on all scales. Fractals may or may not be exactly the same at all scales. The plot of the quantity versus the scale on a log-log graph gives a staright line.The slope of the line gives what is called the fractal dimension. As mathematical equations fractals are not differential anywhere which means that they cannot be measured in any of the traditional ways. There is a certain paradox associated with the property of fractals which implies that the smaller a scale is used for measuring the length of a fractal, the larger the value of the measurement.

There are two types of fractals-*Deterministic and Statisticals*. Some of the deterministic fractals are Cantor Sets, Mandelbrot sets and Koch Curves while Levy Flights come under the category of Statistical fractals. In this work we are going to discuss only about Satistical Fractals (precisely *Levy Flights*).

Since fractal dimensions characterise fractals, in the Chapter-2 we are going to discuss about how to determine them. Three important methods have been discussed here. Chapter-3 has discussions about Levy Flight and its properties. Generating a realisation of Levy Flight numerically is a tedious task and in Chapter-4, I have discussed two such methods for doing the same. Also I have shown how to generate distributions using these two methods. In Chapter-5, I have shown how to numerically determine the fractal dimension. Finally Chapter-6 talks about the conclusions from this analysis.

# Chapter 2

# FRACTAL DIMENSIONS

Fractal Dimensions unlike other dimension can measure the complexity of structures. Measuring the surface area of complex stuctures like the inside of a kidney or the brain or for stuctures such as a cauliflower or brocolli can be more accurately achieved by fractal dimension.Fractal Dimension allows us to measure the degree of complexity by evaluating how fast our measurements increase or decrease as our scale becomes larger or smaller.

Here I am going to discuss about three types of Fractal Dimension Methods- (i) Self Similarity Method, (ii) Box counting method and (iii) the General Method. Before proceeding further on dimension measuring algorithms, it is important to know how a power law works. Essentially, a data set behaves with a power law relationship if it fits the following equation: $y = cx^d$ where c is a constant. One way to determine if data fit a power law relationship is to plot the $log(y)$ versus the $log(x)$. If the plot is a straight line, then it is a power law relationship with slope d.

## 2.1 Self Similarity Method

To Measure the Self-Similar Dimension, the picture must be self-similar. The power law holds and in this case is

$$a = \frac{1}{s^D} \tag{2.1}$$

where $a$ is the number of pieces, $s$ is the reduction factor, and $D$ is the self-similar dimension measure. For example, if a line is broken into three pieces, each is going to be one-third the length of the original. Therefore $a = 3$, $s = (1/3)$, and $D = 1$.

## 2.2 Box-Counting

To calculate the box-counting dimension, we need to place the picture on a grid. The length of a box on the grid is $s = 1/(\text{width of the box})$. For example, if the grid is 150 blocks tall and 150 blocks wide, $s = 1/150$. Then, count the number of blocks that the picture touches. Label this number N(s). Now, resize the grid and repeat the process. Plot the values found on a graph where the x-axis is the log(s) and the y-axis is the log(N(s)). Apply the line of best fit and find the slope. The box-counting dimension measure is equal to the slope of that line. The Box-counting dimension is much more widely used than the self-similarity dimension since the box-counting dimension can measure pictures that are not self-similar (and most real-life applications are not self-similar).

## 2.3 General Method

This method is similar to the Box-Counting Method but is more applicable when there is a statistical distribution of points. This method can be applied to all the three dimensions. In this method we count the number of points $N$ within a region $r$ ($r \to length$ in 1-D, $r \to$ radius of a circle in 2-D and $r \to$ radius of a sphere in 3-D). A Graph of $log(N)$ versus $log(r)$ is plotted.The slope of such a graph then gives the dimension $D$ for the fractal.

For the calculation here, the General Method has been used.

# Chapter 3

# STATISTICAL FRACTALS - LEVY FLIGHTS

Levy Flights are a class of non-Gaussian random processes whose stationary increments are distributed according to the Levy stable distribution. Levy stable laws have the following properties: (i) The probability density of Levy stable law decay in a asymptotic power law with diverging variance thus naturally describing many fluctuating processes with largely scattering statistics. (ii) Levy flights are statistically self-repeatating, a property catering for the description of random fractal processes.

Levy flights are Markovian stochastic processes whose individual step sizes are such that they are are distibuted with the probability density function (PDF) $\lambda(x)$ decaying at large x as,

$$\lambda(x) \backsimeq |x|^{-1-\alpha} \tag{3.1}$$

with $0 < \alpha < 2$. Since the divergence of their variance

$$\langle x^2 \rangle \to \infty \tag{3.2}$$

extreme long jumps may occur and have typical trajectories which are self-similar, on all scales showing clusters of shorter jumps intersparsed by long excursions. The trajectory of Levy Flights possess the fractal dimension given by

$$df = \alpha \tag{3.3}$$

As the Gaussian distribution emerges as the limit distribution of independent identically distributed random variables with finite variance due to the central limit theorem, Levy stable distributions represent the limit distributions of such random variables with diverging variance. In that sense, the Gaussian distribution represents the limiting case of the basin of attraction of the so-called generalized central limit theorem with $\alpha = 2$.

In Fourier Space the Levy Flight distribution is given by

$$f(k) = \exp\left[-i\mu k - \sigma^\alpha |k|^\alpha \left(1 - i\beta \frac{k}{|k|} \varpi(k,\alpha)\right)\right] \tag{3.4}$$

where

$$\varpi = \begin{cases} \tan\frac{\pi\alpha}{2} & \text{if } \alpha \neq 1,\ 0 < \alpha < 2 \\ -\frac{2}{\pi} ln|k| & \text{if } \alpha = 1 \end{cases}$$

Here $\alpha$, $\mu$, $\sigma$ and $\beta$ are the four parameters which are *fractal dimension, mean value, standard deviation* and *skewness* of the distribution respectively. In our approximation later on we will use $\mu = 0$, $\beta = 0$ and$\sigma = 1$ for simplicity. Thus in our approximation [3.4] becomes

$$f(k) = \exp\left[-|k|^\alpha\right] \tag{3.5}$$

Using this value of $f(k)$, the distribution $f(x)$ for different $\alpha$ values can be calculated in real space incorporating the inverse fourier transform.

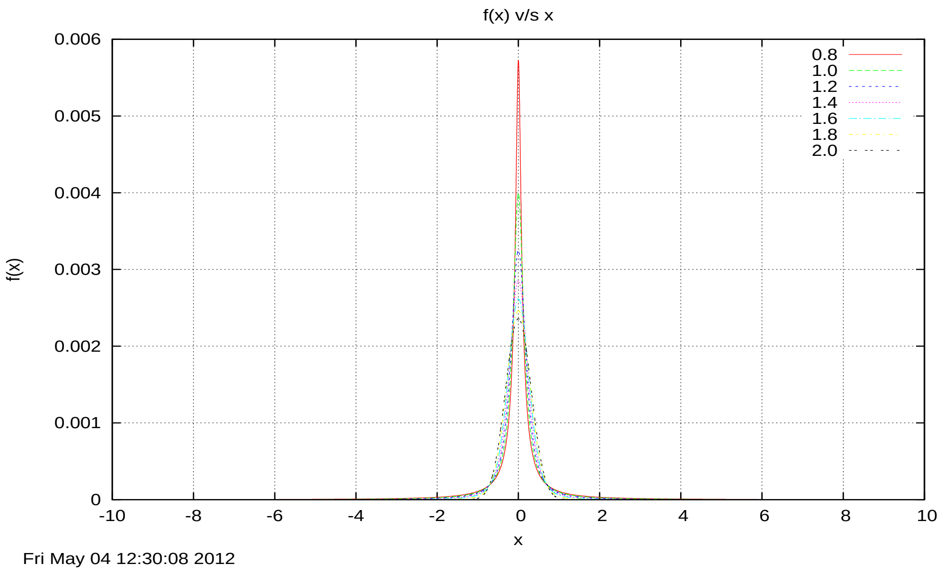


Figure 3.1: Levy Flight Distribution for different $\alpha$ Values

# Chapter 4

# LEVY FLIGHT GENERATION

In this work I have tried to generate random points following a Levy Flight distribution and then see if such a distribution has the characteristic fractal dimension given by $\alpha$. Here I have used two methods for generating random points following a characteristic Levy distribution. The two methods are- (i) The Asymptotic Method and (ii) The Fourier Method. I will now explain each of these methods, the codes used, their advantages and disadvantages.

## 4.1 The Asymptotic Method

We can estimate the behaviour of Levy Flight distributions in real space only asymptotically *i.e.* its behavior for large values of $x$. So the distribution I have accounted for here behaves as,

$$f(x) = A \begin{cases} 1 & 0 < x < 1 \\ \frac{1}{x^{-1-\alpha}} & x > 1 \end{cases}$$

where $A$ is the normalisation factor and on normalising we obtain

$$A = \frac{1+\alpha}{\alpha} \tag{4.1}$$

In order to get a random numbers following this kind of a distribution, we need to generate its cumulative distribution $F(x)$ where

$$F(x) = \int_0^x f(x)dx \tag{4.2}$$

Since the function $f(x)$ is normalised, $F(x)$ has values such that $0 < F(x) < 1$. Thus generating random $F(x)$ and inverting it to obtain the value of the corresponding $x$, we can get a distribution of random points following $f(x)$. Here $f(x)$ are random in the radial co-ordinate. For isotropy values of $\theta$ are generated uniformly such that $0 < \theta < 2\pi$. These are then transformed to the corresponding $x$-$y$ co-ordinates and is generated to give a random walk. To numerically generate such a realisation the code written is shown in [Fig.4.1]. A corresponding graph is also plotted for such a realisation as shown in [Fig.4.2]and [Fig.4.3]. From [Fig.4.2] and [Fig.4.3]we can get

```
%alternate program for levy flight
n=1000000;
alpha=1.2;
p=zeros(n,1);
theta=2.0*pi*rand(n,1);

for i=1:n
        fp=rand(1,1);
        b=((alpha+1.)/alpha)*fp;
        c=2.-b;
        p(i)=(1/c).^(1/alpha);
endfor

x=p.*cos(theta);
y=p.*sin(theta);

xs=zeros(n,1);
ys=zeros(n,1);
xs(1)=x(1);
ys(1)=y(1);

for j=2:n
        xs(j)=xs(j-1)+x(j);
        ys(j)=ys(j-1)+y(j);
endfor
```

Figure 4.1: Program for Levy Flight Distribution using Aymptotic Method

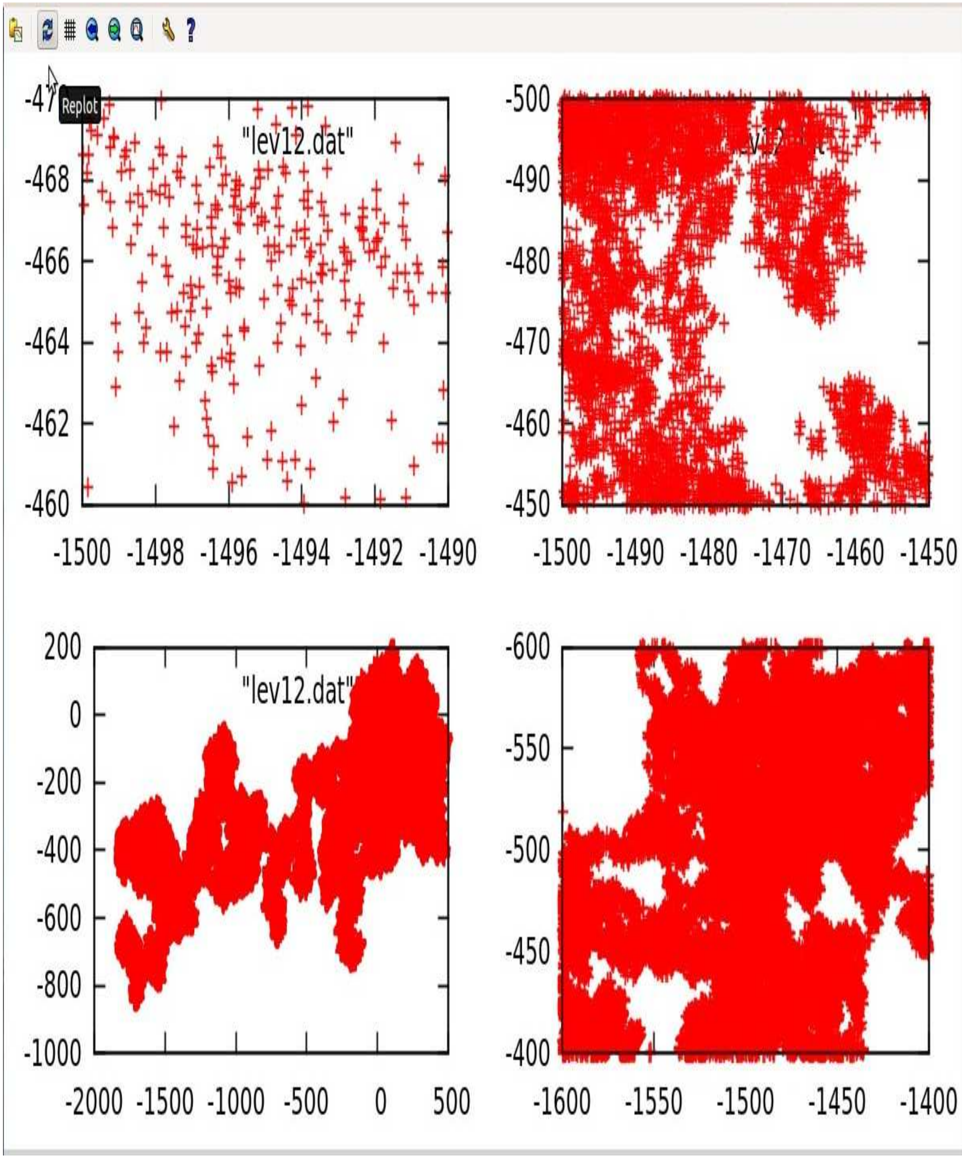


Figure 4.2: Levy Distribution with $\alpha = 1.2$ at different scales

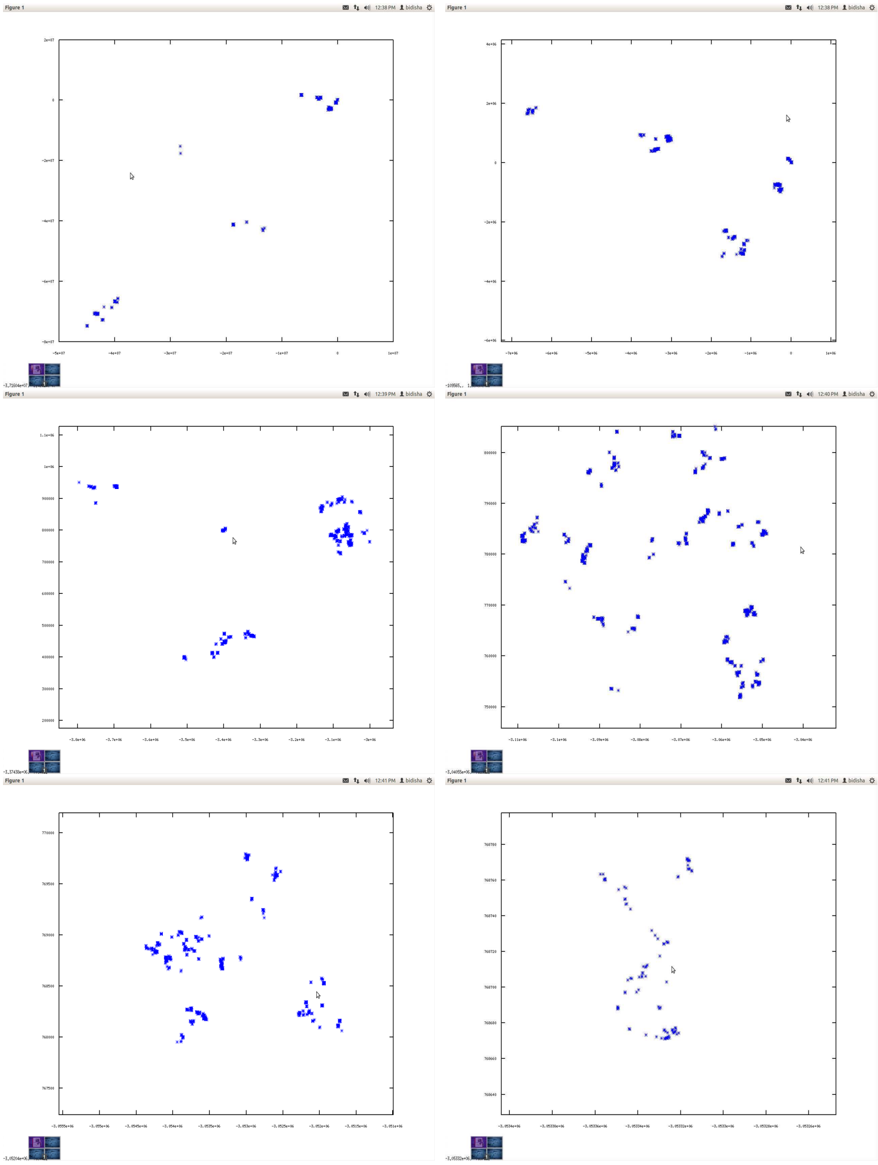

Figure 4.3: Levy Distribution with $\alpha = 0.6$ at different scales

a glimpse of self similarity at different scales. The advantage of this method is that even for smaller values of $\alpha$ (which implies a long tail distribution), we are able to get quite an accurate picture numerically. The only disadvantage in this case is that we cannot get a better picture for $0 < x < 1$. Thus an approximate value of $f(x)$ is assumed in this case.

## 4.2 The Fourier Method

The Fourier Method gives the most apposite picture of a Levy Flight Distribution analytically but since it is not possible to evaluate the Inverse Fourier Transform analytically for all values of $\alpha$, one has to resort to numerical methods for evaluating Inverse Fourier Transform. In this method one takes the fourier transform of the distribution which has the form

$$f(k) = \exp\left[-|k|^{\alpha}\right] \tag{4.3}$$

for $\mu = 0$, $\beta = 0$ and $\sigma = 1$ as discussed earlier. Numerically we cannot evaluate the continuous Fourier Transform but something similar is done which goes by the name of Fast Fourier Transform [FFT] which is a type of Discrete Fourier Transform. Retreiving $f(x)$ from $f(k)$, we then interpolate using the output of the FFT to get $f(x)$ for intermediate values of $x$ and generate a large data file. The normalised cumulative distribution $F(x)$ for $f(x)$ is found. We then generate some random numbers and compare with the generated $F(x)$ to get the corresponding value of $x$. The value of $x$ so generated follows the Levy distribution for the particular value of $\alpha$. The rest follows from the steps used in the Asymptotic Method. The main code and the subroutines are mentioned in [Fig.4.4], [Fig.4.5], [Fig.4.6] and [Fig.4.7]. The main disadvantage of using this method is that due to the limitations of generating an appropriate fast fourier tranform and interpolated values, it gives erronious results for small values of $\alpha$ where the long tail is cut off. Even though I have generated a random walk using this method but the results have not been further used in characterising Levy distribution by its dimensional analysis. Also in this method we need to generate a larger data set compared to the number of random points we require for the realisation of the distribution. If we can improvise upon the calculation of FFT and Interpolations, this method would have been the best is generating a Levy Distribution.

```
%program for levyflight distribution

alpha=0.8;
n=10000;
t=20*[-n/2:(n/2)-1]/n;
ft=levyinf(alpha,n);

levy(:,1)=t;levy(:,(i+1))=ft;
save levy.dat levy
alpha=alpha+0.2;

xg=[0:0.00001:3];
fxg=levyint(xg);

y0=0.00000001;
y=lsode("levyinteg",y0,xg);

z=y./(max(y));

p=zeros(10000,1);
theta=2.0*pi*rand(10000,1);
acc=0.0001;

for i=1:10000
        k=rand(1,1);
        for j=1:300001
                l=z(j)-k;
                if(l<=acc)
                        p(i)=xg(j);
                endif
        endfor
endfor

x1=p.*cos(theta);
y1=p.*sin(theta);

xs=zeros(10000,1);
ys=zeros(10000,1);
xs(1)=x1(1);
ys(1)=y1(1);

for j=2:10000
        xs(j)=xs(j-1)+x1(j);
        ys(j)=ys(j-1)+y1(j);
endfor
```

Figure 4.4: Program for Levy Flight Distribution using Fourier Method

```
function dist=levyinf(alpha,n)
        k=linspace(-50,50,1000);
        fk=(sqrt(gamma((1.0/alpha)+1)))*exp(-((abs(k)).^alpha)/2);
        dist=abs(fftshift(ifft(fk,n)));
endfunction
```

Figure 4.5: The function 'levyinf' to calculate Inverse Fourier Transform

```
function fun=levyint(xg)
        load levy.dat
        fun=interp1(levy(:,1),levy(:,2),xg,'extrap');
endfunction
```

Figure 4.6: The function 'levyint' for interpolation

```
function ydot=levyinteg(y,xg);
        ydot=levyint(xg);
endfunction
```

Figure 4.7: The function 'levyinteg' for generating cumulative distribution

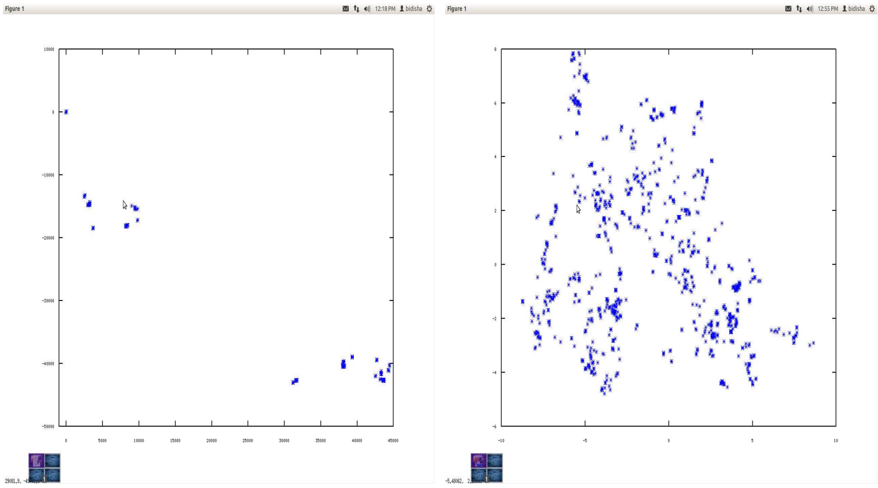

Figure 4.8: Comparison between Analytic Method and Fourier Method

In order to compare the distribution of these methods, both the generated distribution is shown in [Fig.4.8]. The distribution on the left figure is generated by analytic method while that on the right is generated by Fourier Method both using the $\alpha = 0.6$. From this we can infer that since the Fourier Method chopps off the long tail of the distribution, it appears more uniformly distributed around a mean value and hence provides erronious results.

# Chapter 5

# DETERMINATION OF DIMENSION

To Determine the fractal dimension of Levy Flight, I have used the General Method Of Chapter.2. This method is done for some thousand random points and then averaged over to get an accurate data set. The octave code used is given in [Fig.5.1]. A graph of $log(n)$ versus $log(r)$ is plotted. A straight line is fitted to the graph using gnuplot which is shown in [Fig.5.2]. The slope of the fitted straight line comes out to be $m = 1.42$. This is the determined value of alpha with a relative error of 18.33%. The output seems reasonably fair compared to the given value $\alpha = 1.20$. Thus from this analysis one can infer that the analytic method gives a good realisation to a Levy Flight distribution.

```
%program for fractal analysis

load lev12.dat

dr=5;
m=zeros(1000,40);
for l=1:1000
        n=int64(100000*rand(1,1));
        xc=lev12(n,1);
        yc=lev12(n,2);
        r=5;
        for k=1:40
                for i=1:100000;
                        x=(lev12(i,1)-xc);
                        y=(lev12(i,2)-yc);
                        rc=abs(sqrt((x.^2)+(y.^2)));
                        if (rc <= r)
                                m(l,k)=m(l,k)+1;
                        endif
                endfor
                r=r+dr;
        endfor
endfor
```

Figure 5.1: Program for determining the fractal dimension

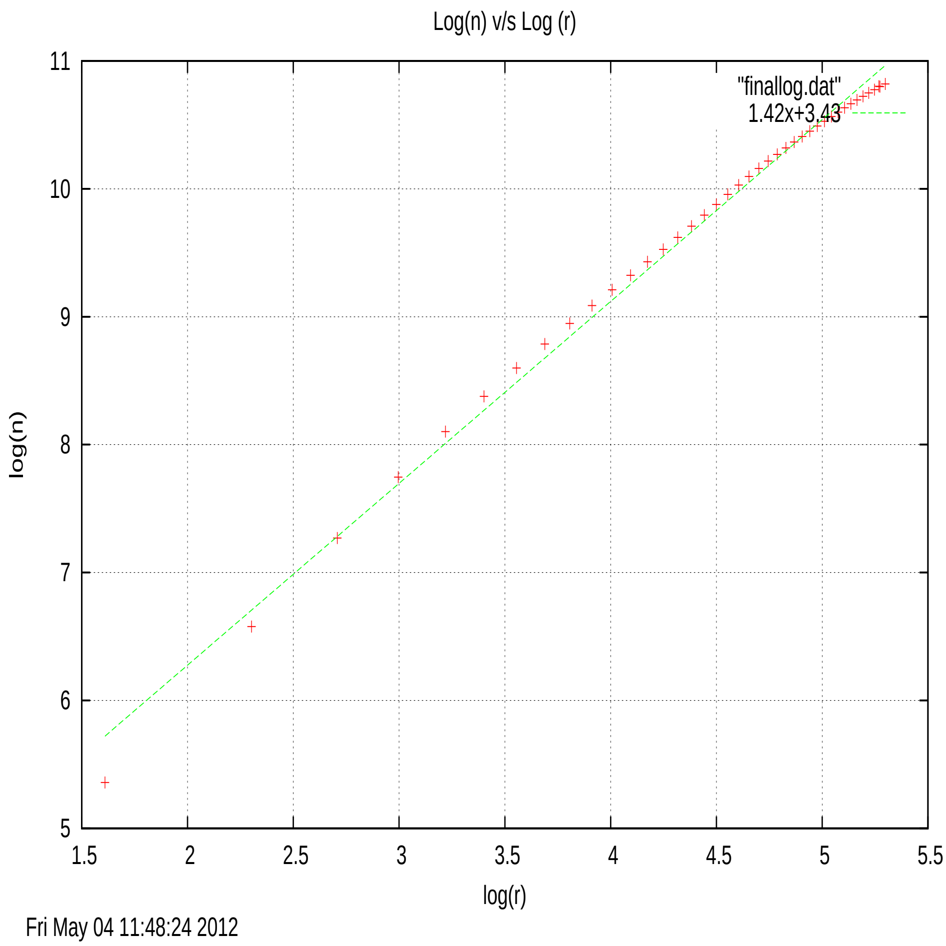


Figure 5.2: Plot of $log(n)$ versus $log(r)$

# Chapter 6

# CONCLUSION

Statistical Fractals like Levy Flights have a lot of important applications like studying the clustering of galaxies at large scales, studying the spread of some disease, the photon path in turbulent fluid and many other such things which we come accross in our day to day life and which affect our life in some important way or the other. Thus studying and characterising them is very important. Analytical results can give us some approximations but for a better and lucid picture we have to resort to numerical methods. A few of such methods have been presented here.

From all the above analysis, we can see that generating a Statistical distribution numerically does not give accurate results because of its own limitations and approximations but the accuracy can be improved by better techniques. Here we have shown such improvent using two different methods. The accuracy of the method can be quantified which here can be found by calculating the fractal dimension.

Since the Fourier method from its distribution itself looked erronious, further analysis using it has not been done. Instead I resorted to the analytical method which calculates $\alpha = 1.42$ instead of the accurate value of $\alpha = 1.20$ which seems reasonable enough.

Finally I would like to say that more such methods could be there or one can improve upon the existing ones to get fair results.

# Bibliography


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