TO GET SCHWARZSCHILD

BLACKHOLE SOLUTION USING

MATHEMATICA

PROJECT REPORT FOR COMPULSORY COURSE

WORK PAPER PHY 601

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INTRODUCTION:

When faced with a difficult set of mathematical equations, the first course of action one often takes is to look for special cases that are the easiest to solve. It turns out that such an approach often yields insights into the most interesting and physically relevant situations. This is as true for general relativity as it is for any other theory of mathematical physics.

Therefore for our first application of General theory of Relativity, we consider a solution to the field equations that is time independent and spherically symmetric. Such a scenario can describe the gravitational field found outside of the Sun, for example. Since we might be interested only in the field outside of the matter distribution, we can simplify things even further by restricting our attention to the matter-free regions of space in the vicinity of some mass. Within the context of relativity, this means that one can find a solution to the problem using the vacuum equations and ignore the stress-energy tensor.

The solution we will obtain is known as *Schwarzschild Solution*. It was found in 1916 by the German Physicist Karl Schwarzschild while he was serving on the Russian front during the first world war.

In this report I have tried to get the Schwarzschild solution with the help of Mathematica software. If someone has tried earlier to get the solution by other methods, he can easily find that knowing Mathematica applications how easier it is than the other methods. Actually, using *Mathematica* we can easily solve tedious problems of Theoretical Physics.

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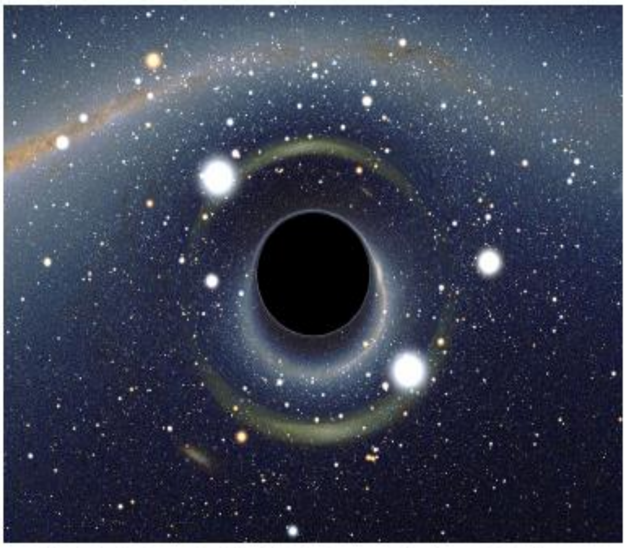

Simulated view of a black hole (center) in front of the Large Magellanic Cloud. Note the gravitational lensing effect, which produces two enlarged but highly distorted views of the Cloud. Across the top, the Milky Way disk appears distorted into an arc.

BLACK HOLE:

A black hole is a region of spacetime from which nothing, not even light, can escape. The theory of general relativity predicts that a sufficiently compact mass will deform spacetime to form a black hole. Around a black hole there is a mathematically defined surface called an event horizon that marks the point of no return. It is called "black" because it absorbs all the light that hits the horizon, reflecting nothing, just like a perfect black body in thermodynamics. Quantum mechanics predicts that black holes emit radiation like a black body with a finite temperature. This temperature is inversely proportional to the mass of the black hole, making it difficult to observe this radiation for black holes of stellar mass or greater.

Objects whose gravity field is too strong for light to escape were first considered in the 18th century by John Michell and Pierre-Simon Laplace. The first modern solution of general relativity that would characterize a black hole was found by Karl Schwarzschild in 1916, although its interpretation as a region of space from which nothing can escape was not fully appreciated for another four decades. Long considered a mathematical curiosity, it was during the 1960s that theoretical work showed black holes were a generic prediction of general relativity. The discovery of neutron stars sparked interest in gravitationally collapsed compact objects as a possible astrophysical reality.

Black holes of stellar mass are expected to form when very massive stars collapse at the end of their life cycle. After a black hole has formed it can continue to grow by absorbing mass from its surroundings. By absorbing other stars and merging with other black holes, supermassive black holes of millions of solar masses may form. There is general consensus that supermassive black holes exist in the centers of most galaxies. In particular, there is strong evidence of a black hole of more than 4 million solar masses at the center of our galaxy, the Milky Way.

Despite its invisible interior, the presence of a black hole can be inferred through its interaction with other matter and with light and other electromagnetic radiation. From stellar movement, the mass and location of an invisible companion object can be calculated; in a number of cases the only known object capable of meeting these criteria is a black hole. Astronomers have identified numerous stellar black hole candidates in binary systems by studying the movement of their companion stars in this way.

Physical properties:

The simplest black holes have mass but neither electric charge nor angular momentum. These black holes are often referred to as Schwarzschild black holes after Karl Schwarzschild who discovered this solution in 1916. According to Birkhoff's theorem, it is the only vacuum solution that is spherically symmetric. This means that there is no observable difference between the gravitational field of such a black hole and that of any other spherical object of the same mass. The popular notion of a black hole "sucking in everything" in its surroundings is therefore only correct near a black hole's horizon; far away, the external gravitational field is identical to that of any other body of the same mass.

Solutions describing more general black holes also exist. Charged black holes are described by the Reissner–Nordström metric, while the Kerr metric describes a rotating black hole. The most general stationary black hole solution known is the Kerr–Newman metric, which describes a black hole with both charge and angular momentum.

While the mass of a black hole can take any positive value, the charge and angular momentum are constrained by the mass. In Planck units, the total electric charge Q and the total angular momentum J are expected to satisfy for a black hole of mass M. Black holes saturating this inequality are called extremal. Solutions of Einstein's equations that violate this inequality exist, but they do not possess an event horizon. These solutions have so-called naked singularities that can be observed from the outside, and hence are deemed unphysical. The cosmic censorship hypothesis rules out the formation of such singularities, when they are created through the gravitational collapse of realistic matter. This is supported by numerical simulations.

Due to the relatively large strength of the electromagnetic force, black holes forming from the collapse of stars are expected to retain the nearly neutral charge of the star. Rotation, however, is expected to be a common feature of compact objects. The black-hole candidate binary X-ray source GRS 1915+105 appears to have an angular momentum near the maximum allowed value.

Black hole classifications

Class	Mass	Size
Supermassive black hole	~105–109 MSun	~0.001–10 AU
Intermediate-mass black hole	~103 MSun	~103km =REarth
Stellar black hole	~10 MSun	~30 km
Micro black hole	up to ~MMoon	up to ~0.1 mm

Black holes are commonly classified according to their mass, independent of angular momentum J or electric charge Q. The size of a black hole, as determined by the radius of the event horizon, or Schwarzschild radius, is roughly proportional to the mass M through

$$r_{sh} = 2GM/c^2 \sim 2.95\ M/M_{Sun}\ \text{km},$$

Where r_{sh} is the Schwarzschild radius and M_{Sun} is the mass of the Sun. This relation is exact only for black holes with zero charge and angular momentum; for more general black holes it can differ up to a factor of 2.

Singularity:

At the center of a black hole as described by general relativity lies a gravitational singularity, a region where the spacetime curvature becomes infinite. For a non-rotating black hole this region takes the shape of a single point and for a rotating black hole it is smeared out to form a ring singularity lying in the plane of rotation. In both cases the singular region has zero volume. It can also be shown that the singular region contains all the mass of the black hole solution. The singular region can thus be thought of as having infinite density.

Observers falling into a Schwarzschild black hole (i.e. non-rotating and no charges) cannot avoid being carried into the singularity, once they cross the event horizon. They can prolong the experience by accelerating away to slow their descent, but only up to a point; after attaining a certain ideal velocity, it is best to free fall the rest of the way. When they reach the singularity, they are crushed to infinite density and their mass is added to the total of the black hole. Before that happens, they will have been torn apart by the growing tidal forces in a process sometimes referred to as spaghettification or the noodle effect. In the case of a charged (Reissner–Nordström) or rotating (Kerr) black hole it is possible to avoid the singularity.

Extending these solutions as far as possible reveals the hypothetical possibility of exiting the black hole into a different spacetime with the black hole acting as a wormhole. The possibility of traveling to another universe is however only theoretical, since any perturbation will destroy this possibility. It also appears to be possible to follow closed timelike curves (going back to one's own past) around the Kerr singularity, which lead to problems with causality like the grandfather paradox. It is expected that none of these peculiar effects would survive in a proper quantum mechanical treatment of rotating and charged black holes.

The appearance of singularities in general relativity is commonly perceived as signaling the breakdown of the theory. This breakdown, however, is expected; it occurs in a situation where quantum mechanical effects should describe these actions due to the extremely high density and therefore particle interactions. To date it has not been possible to combine quantum and gravitational effects into a single theory. It is generally expected that a theory of quantum gravity will feature black holes without singularities.

Schwarzschild Solution:

As a simple application of Einstein’s equations, let us determine the gravitational field (metric) of a static, spherically symmetric star. Many stars conform to this condition. There are also many others that behave differently. For example, a star may have asymmetries associated with it, it may be rotating or it may be pulsating. However, the static, spherically symmetric star is a simple example for which the metric can be solved exactly. Therefore, it leads to theoretical predictions which can be verified as tests of general relativity.

Line element

Although Einstein's equations are highly nonlinear, the reason why we can solve them for a static, spherically symmetric star is that the symmetry present in the problem restricts the form of the solution greatly. For example, since the gravitating mass (source) is static, the metric components would be independent of time. Furthermore, the spherical symmetry of the problem requires that the components of the metric can depend only on the radial coordinate r. Let us recall that in spherical coordinates, the flat space-time can be characterized by the line element:

$$d\tau^2 = dt^2 - (dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)) \tag{1}$$

We can generalize this line element to a static, isotropic curved space as

$$d\tau^2 = A(r)dt^2 - (B(r)dr^2 + C(r)r^2 d\theta^2 + D(r)r^2 \sin^2\theta d\varphi^2) \tag{2}$$

The following assumptions have gone into writing the line element in this form. First of all since the metric components are independent of time, the line element should be invariant if we let $dt \to -dt$. This implies that linear terms in dt cannot occur. Isotropy similarly tells that if we let $d\theta \to -d\theta$ or $d\varphi \to -d\varphi$, the line element should be invariant. Thus terms of the form $drd\theta$, $drd\varphi$ or $d\theta d\varphi$ cannot occur either. This restricts the form of the metric to be diagonal.

Let us now look at the line element (2) at a fixed time and radius. At the north pole ($d\varphi = 0$) with $\epsilon = rd\theta$, we have

$$d\tau^2 = -C(r)\, \epsilon^2\,. \tag{3}$$

On the other hand, if we look at the line element in the same slice of space-time but

at the equator ($\theta = \pi$) with $\epsilon = rd\varphi$, then

$$d\tau^2 = -D(r)\, \epsilon^2\,. \tag{4}$$

However, if the space is isotropic then these two lengths must be equal which requires

$$C(r) = D(r). \tag{5}$$

Thus we can write the line element (2) as

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - C(r)r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{6}$$

We note here that the function C(r) in (6) is redundant in the sense that it can be scaled away. Namely, if we let

$$r \to r^{\sim} = [C(r)]^{1/2} r, \tag{7}$$

then

$$dr^{\sim} = dr\, [(C(r))^{1/2} + rC'(r)/2(C(r))^{1/2}]$$

$$\text{or} \qquad dr = f(r^{\sim})\, dr^{\sim} \tag{8}$$

where we have identified (prime denotes a derivative with respect to r)

$$f(r^{\sim}) = 2(C(r))^{1/2} / 2C(r)+rC'(r) \tag{9}$$

This shows that with a proper choice of the coordinate system the line element for a static, spherically symmetric gravitational field can be written as

$$d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \tag{10}$$

which is known as the general Schwarzschild line element.

Connection

As we see now, the Schwarzschild line element (10) is given in terms of two unknown functions A(r) and B(r). The metric components can be read off from the line element (10) to be

$g_{00} = g_{tt} = A(r)$,

$g_{11} = g_{rr} = -B(r)$,

$g_{22} = g_{\theta\theta} = -r^2$,

$g_{33} = g_{\varphi\varphi} = -r^2 \sin^2\theta$. (11)

This is a diagonal metric and hence the nontrivial components of the inverse metric can also be easily written down as

$g^{00} = g^{tt} = 1/A(r)$,

$g^{11} = g^{rr} = -1/B(r)$,

$g^{22} = g^{\theta\theta} = -1/r^2$,

$g^{33} = g^{\phi\phi} = -1/r^2 \sin^2\theta$. (12)

We can solve Einstein's equations far away from the star to determine the forms of the functions A(r), B(r). That is outside the star we can solve the empty space equation

$$R_{\mu\nu} = 0, \tag{13}$$

Subject to the boundary condition that infinitely far away from the star, the metric reduces to Minkowski form (8.1). To solve Einstein's equations we must, of course, calculate the connections and the curvature tensor. For example, the definition of the Christoffel symbol we have

$$`\Gamma^{\mu}{}_{\nu\lambda} = -(1/2)\ g^{\mu\rho} (\partial_{\nu} g_{\lambda\rho} + \partial_{\lambda} g_{\rho\nu} - \partial_{\rho} g_{\nu\lambda}), \qquad (14)$$

And since we know the metric components, these can be calculated. But this method is tedious and let us try to determine the components of the connection using Mathematica software package.

Christoffel Symbols and Geodesic Equation

This is a *Mathematica* program to compute the Christoffel and the geodesic equations, starting from a given metric $g_{\alpha\beta}$. The Christoffel symbols are calculated from the formula

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\, g^{\lambda\sigma}\left(\partial_\mu\, g_{\sigma\nu} + \partial_\nu\, g_{\sigma\mu} - \partial_\sigma\, g_{\mu\nu}\right)$$

where $g^{\lambda\sigma}$ is the matrix inverse of $g_{\lambda\sigma}$ called the inverse metric. This is the solution of the relation (8.19) and the notation for the inverse metric is standard [cf (20.17)]. The components of the geodesic equation are

$$du^{\alpha}/d\tau = -\Gamma^{\alpha}{}_{\beta\gamma}\, u^{\beta} u^{\gamma} \;.$$

You must input the covariant components of the metric tensor $g_{\mu\nu}$ by editing the relevant input line in this *Mathematica* notebook. You may also wish to change the names of the coordinates. The nonzero components of the above quantities are displayed as the output.

■ Clearing the values of symbols:

First clear any values that may already have been assigned to the names of the various objects to be calculated. The names of the coordinates that you will use are also cleared.

```
Clear[coord, metric, inversemetric, affine, r, θ, ϕ, t]
```

■ Setting The Dimension

The dimension **n** of the spacetime (or space) must be set:

```
n = 4
```

```
4
```

■ Defining a list of coordinates:

The example given here is the wormhole metric (7.40). Note that for convenience t is denoted by x^4 rather than x^0 and summations run from 1 to 4 rather than 0 to 3.

```
coord = {r, θ, ϕ, t}
```

```
{r, θ, ϕ, t}
```

You can change the names of the coordinates by simply editing the definition of **coord**, for example, to **coord = {x, y, z, t}**, when another set of coordinate names is more appropriate.

■ Defining the metric:

Input the metric as a list of lists, i.e., as a matrix. You can input the components of any metric here, but you must specify them as explicit functions of the coordinates.

```
metric = {{1, 0, 0, 0}, {0, r^2 + b^2, 0, 0}, {0, 0, (r^2 + b^2) Sin[θ]^2, 0}, {0, 0, 0, -1}}
```

$$\{\{1, 0, 0, 0\}, \{0, b^2 + r^2, 0, 0\}, \{0, 0, (b^2 + r^2)\, \text{Sin}[\theta]^2, 0\}, \{0, 0, 0, -1\}\}$$

You can also display this in matrix form:

```
metric // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b^2 + r^2 & 0 & 0 \\ 0 & 0 & (b^2 + r^2)\, \text{Sin}[\theta]^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

▪ Note:

It is important not to use the symbols, **i, j, k, l, n,** or **s** as constants or coordinates in the metric that you specify above. The reason is that the first four of those symbols are used as summation or table indices in the calculations done below. The last is the dimension of the space.

▪ Calculating the inverse metric:

The inverse metric is obtained through matrix inversion.

```
inversemetric = Simplify[Inverse[metric]]
```

$$\left\{\{1, 0, 0, 0\}, \left\{0, \frac{1}{b^2 + r^2}, 0, 0\right\}, \left\{0, 0, \frac{\text{Csc}[\theta]^2}{b^2 + r^2}, 0\right\}, \{0, 0, 0, -1\}\right\}$$

The inverse metric can also be displayed in matrix form:

```
inversemetric // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{b^2+r^2} & 0 & 0 \\ 0 & 0 & \frac{\text{Csc}[\theta]^2}{b^2+r^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

▪ Calculating the affine connection:

The calculation of the components of the affine connection is done by transcribing the definition given earlier into the notation of *Mathematica* and using the *Mathematica* functions **D** for taking partial derivatives, **Sum** for summing over repeated indices, **Table** for forming a list of components, and **Simplify** for simplifying the result.

```
affine := affine = Simplify[Table[(1 / 2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]] ] +
          D[metric[[s, k]], coord[[j]] ] - D[metric[[j, k]], coord[[s]] ]), {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}] ]
```

■ Displaying the affine connection:

The nonzero components of the affine connections are displayed below. You need not follow the details of constructing the functions that we use for that purpose. Because the affine connection is symmetric under interchange of the last two indices, only the independent components are displayed.

```
listaffine := Table[If[UnsameQ[affine[[i, j, k]], 0],
   {ToString[Γ[i, j, k]], affine[[i, j, k]]}] , {i, 1, n}, {j, 1, n}, {k, 1, j}]

TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2], TableSpacing→{2, 2}]
```

Γ[1, 2, 2]	$-r$
Γ[1, 3, 3]	$-r\,\mathrm{Sin}[\theta]^2$
Γ[2, 2, 1]	$\frac{r}{b^2+r^2}$
Γ[2, 3, 3]	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
Γ[3, 3, 1]	$\frac{r}{b^2+r^2}$
Γ[3, 3, 2]	$\mathrm{Cot}[\theta]$

■ Calculating the geodesic equations:

The geodesic equations are calculated by asking *Mathematica* to carry out the sum $-\Gamma^{\alpha}{}_{\beta\gamma}u^{\beta}u^{\gamma}$, where u^{α} are the components of the four–velocity. (This gives the derivitive of u^{α} with respect to proper time τ. (This is replaced by s if the geodesics are spacelike.)

```
geodesic := geodesic = Simplify[Table[-Sum[affine[[i, j, k]] u[j] u[k], {j, 1, n},
      {k, 1, n}], {i, 1, n}]]
```

■ Displaying the geodesic equations:

```
listgeodesic := Table[{"d/dτ" ToString[u[i]], "=", geodesic[[i]]}, {i, 1, n}]

TableForm[listgeodesic, TableSpacing→{2}]
```

d/dτ u[1]	=	$r\,(u[2]^2+\mathrm{Sin}[\theta]^2\,u[3]^2)$
d/dτ u[2]	=	$-\frac{2\,r\,u[1]\,u[2]}{b^2+r^2}+\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,u[3]^2$
d/dτ u[3]	=	$-\frac{2\,r\,u[1]\,u[3]}{b^2+r^2}-2\,\mathrm{Cot}[\theta]\,u[2]\,u[3]$
d/dτ u[4]	=	0

■ Acknowledgment

This program was adapted from the notebook *Curvature and the Einstein* equation kindly written by *Leonard Parker* especially for this text.

PDF version of subprograms run into Mathematica to get Schwarzschild blackhole solution have been shown in the next page.

```
Clear[coord, metric, inversemetric, affine, t, r, θ, ϕ]

n = 4

4

coord = {t, r, θ, ϕ}

{t, r, θ, ϕ}

metric = {{-A[r], 0, 0, 0}, {0, B[r], 0, 0}, {0, 0, r ^2, 0}, {0, 0, 0, (r ^2) Sin[θ] ^2}}
```

$$\left\{\{-A[r], 0, 0, 0\}, \{0, B[r], 0, 0\}, \left\{0, 0, r^2, 0\right\}, \left\{0, 0, 0, r^2 \operatorname{Sin}[\theta]^2\right\}\right\}$$

```
metric // MatrixForm
```

$$\begin{pmatrix} -A[r] & 0 & 0 & 0 \\ 0 & B[r] & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \operatorname{Sin}[\theta]^2 \end{pmatrix}$$

```
inversemetric = Simplify[Inverse[metric]]
```

$$\left\{\left\{-\frac{1}{A[r]}, 0, 0, 0\right\}, \left\{0, \frac{1}{B[r]}, 0, 0\right\}, \left\{0, 0, \frac{1}{r^2}, 0\right\}, \left\{0, 0, 0, \frac{\operatorname{Csc}[\theta]^2}{r^2}\right\}\right\}$$

```
affine := affine = Simplify[Table[(1/2) * Sum[(inversemetric[[i, s]]) *
        (D[metric[[s, j]], coord[[k]] ] +
          D[metric[[s, k]], coord[[j]] ] - D[metric[[j, k]], coord[[s]] ]), {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}] ]

listaffine := Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[i, j, k]], affine[[i, j, k]]}] ,
  {i, 1, n}, {j, 1, n}, {k, 1, j}]

TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2], TableSpacing → {2, 2}]
```

Γ[1, 2, 1]	$\frac{A'[r]}{2 A[r]}$
Γ[2, 1, 1]	$\frac{A'[r]}{2 B[r]}$
Γ[2, 2, 2]	$\frac{B'[r]}{2 B[r]}$
Γ[2, 3, 3]	$-\frac{r}{B[r]}$
Γ[2, 4, 4]	$-\frac{r \operatorname{Sin}[\theta]^2}{B[r]}$
Γ[3, 3, 2]	$\frac{1}{r}$
Γ[3, 4, 4]	$-\operatorname{Cos}[\theta] \operatorname{Sin}[\theta]$
Γ[4, 4, 2]	$\frac{1}{r}$
Γ[4, 4, 3]	$\operatorname{Cot}[\theta]$

```
geodesic := geodesic = Simplify[Table[-Sum[affine[[i, j, k]] u[j] u[k], {j, 1, n},
      {k, 1, n}], {i, 1, n}]]

listgeodesic := Table[{"d/dτ" ToString[u[i]], "=", geodesic[[i]]}, {i, 1, n}]
```

```
TableForm[listgeodesic, TableSpacing → {2}]
```

$$d/d\tau\ u[1] = -\frac{u[1]\ u[2]\ A'[r]}{A[r]}$$

$$d/d\tau\ u[2] = -\frac{-2\ r\ (u[3]^2 + Sin[\theta]^2\ u[4]^2) + u[1]^2\ A'[r] + u[2]^2\ B'[r]}{2\ B[r]}$$

$$d/d\tau\ u[3] = -\frac{2\ u[2]\ u[3]}{r} + Cos[\theta]\ Sin[\theta]\ u[4]^2$$

$$d/d\tau\ u[4] = -\frac{2\ (u[2] + r\ Cot[\theta]\ u[3])\ u[4]}{r}$$

```
riemann := riemann = Simplify[Table[
    D[affine[[i, j, l]], coord[[k]] ] - D[affine[[i, j, k]], coord[[l]] ] +
     Sum[affine[[s, j, l]] affine[[i, k, s]] - affine[[s, j, k]] affine[[i, l, s]],
      {s, 1, n}],
    {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}] ]

listriemann :=
 Table[If[UnsameQ[riemann[[i, j, k, l]], 0], {ToString[R[i, j, k, l]], riemann[[i, j, k, l]]}] ,
  {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, k - 1}]

TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2], TableSpacing → {2, 2}]
```

$$R[1, 2, 2, 1] \quad -\frac{A[r]\ A'[r]\ B'[r] + B[r]\ (A'[r]^2 - 2\ A[r]\ A''[r])}{4\ A[r]^2\ B[r]}$$

$$R[1, 3, 3, 1] \quad \frac{r\ A'[r]}{2\ A[r]\ B[r]}$$

$$R[1, 4, 4, 1] \quad \frac{r\ Sin[\theta]^2\ A'[r]}{2\ A[r]\ B[r]}$$

$$R[2, 1, 2, 1] \quad -\frac{A[r]\ A'[r]\ B'[r] + B[r]\ (A'[r]^2 - 2\ A[r]\ A''[r])}{4\ A[r]\ B[r]^2}$$

$$R[2, 3, 3, 2] \quad -\frac{r\ B'[r]}{2\ B[r]^2}$$

$$R[2, 4, 4, 2] \quad -\frac{r\ Sin[\theta]^2\ B'[r]}{2\ B[r]^2}$$

$$R[3, 1, 3, 1] \quad \frac{A'[r]}{2\ r\ B[r]}$$

$$R[3, 2, 3, 2] \quad \frac{B'[r]}{2\ r\ B[r]}$$

$$R[3, 4, 4, 3] \quad -\frac{(-1 + B[r])\ Sin[\theta]^2}{B[r]}$$

$$R[4, 1, 4, 1] \quad \frac{A'[r]}{2\ r\ B[r]}$$

$$R[4, 2, 4, 2] \quad \frac{B'[r]}{2\ r\ B[r]}$$

$$R[4, 3, 4, 3] \quad 1 - \frac{1}{B[r]}$$

```
ricci := ricci = Simplify[Table[Sum[riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}] ]

listricci :=
 Table[If[UnsameQ[ricci[[j, l]], 0], {ToString[R[j, l]], ricci[[j, l]]}] , {j, 1, n}, {l, 1, j}]
```

```
TableForm[Partition[DeleteCases[Flatten[listricci], Null], 2], TableSpacing → {2, 2}]
```

R[1, 1]	$-\frac{A'[r]\, B'[r]}{4\, B[r]^2} + \frac{\frac{A'[r]}{r} - \frac{A'[r]^2}{4\, A[r]} + \frac{A''[r]}{2}}{B[r]}$
R[2, 2]	$\frac{A[r]\, (4\, A[r] + r\, A'[r])\, B'[r] + r\, B[r]\, (A'[r]^2 - 2\, A[r]\, A''[r])}{4\, r\, A[r]^2\, B[r]}$
R[3, 3]	$\frac{1}{2}\left(2 - \frac{2 + \frac{r\, A'[r]}{A[r]}}{B[r]} + \frac{r\, B'[r]}{B[r]^2}\right)$
R[4, 4]	$\frac{\mathrm{Sin}[\theta]^2\, (-r\, B[r]\, A'[r] + A[r]\, (-2\, B[r] + 2\, B[r]^2 + r\, B'[r]))}{2\, A[r]\, B[r]^2}$

```
scalar = Simplify[Sum[inversemetric[[i, j]] ricci[[i, j]], {i, 1, n}, {j, 1, n}] ]
```

$$\left(r^2\, B[r]\, A'[r]^2 + 4\, A[r]^2\, (-B[r] + B[r]^2 + r\, B'[r]) + r\, A[r]\, (r\, A'[r]\, B'[r] - 2\, B[r]\, (2\, A'[r] + r\, A''[r]))\right) \Big/ \left(2\, r^2\, A[r]^2\, B[r]^2\right)$$

```
einstein := einstein = Simplify[ricci - (1 / 2) scalar * metric]

listeinstein := Table[
  If[UnsameQ[einstein[[j, 1]], 0], {ToString[G[j, 1]], einstein[[j, 1]]}] , {j, 1, n}, {1, 1, j}]

TableForm[Partition[DeleteCases[Flatten[listeinstein], Null], 2], TableSpacing → {2, 2}]
```

G[1, 1]	$\frac{A[r]\, (-B[r] + B[r]^2 + r\, B'[r])}{r^2\, B[r]^2}$
G[2, 2]	$\frac{A[r] - A[r]\, B[r] + r\, A'[r]}{r^2\, A[r]}$
G[3, 3]	$\frac{r\, (-r\, B[r]\, A'[r]^2 - 2\, A[r]^2\, B'[r] + A[r]\, (-r\, A'[r]\, B'[r] + 2\, B[r]\, (A'[r] + r\, A''[r])))}{4\, A[r]^2\, B[r]^2}$
G[4, 4]	$\frac{r\, \mathrm{Sin}[\theta]^2\, (-r\, B[r]\, A'[r]^2 - 2\, A[r]^2\, B'[r] + A[r]\, (-r\, A'[r]\, B'[r] + 2\, B[r]\, (A'[r] + r\, A''[r])))}{4\, A[r]^2\, B[r]^2}$

so that the 11-component of (13) leads to

$R_{11} = 0$

$\Rightarrow R_{11} = A''/2B - A'B'/2B^2 - A'/2B(A'/2A\ B'/2B - 2/r) = 0.$

or,

$$A''/2B - (A'/4B)\,(A'/A + B'/B) + A'/rB = 0. \qquad (15)$$

The 22- component of (13) leads to

$R_{22} = 0$,

$$\text{or,} \quad A''/2A - (1/4)\,(A'/A)(A'/A + B'/B) - B'/rB = 0 \qquad (16)$$

The 33- component of (13) yields

$R_{33} = 0$,

or, $(1/B) + (r/2B)\,(A'/A - B'/B) - 1 = 0$

Multiplying (15) by B/A and subtracting from (16) we have

$$\lim_{r \to \infty} A(r) \longrightarrow 1$$

$$\lim_{r \to \infty} B(r) \longrightarrow 1 \qquad (17)$$

This therefore. Determines the constant of integration in to be

$$k = 1, \qquad (18)$$

and we have

$A(r)\,B(r) = 1$

$$\text{or, } B(r) = 1/A(r) \qquad (19)$$

If we now substitute this relation , we obtain

$$A(r) + (rA/2)\,(A'/A + A'/A) - 1 = 0$$

$$\text{or,} \quad A(r) + rA'(r) = 1$$

$$\text{or,} \quad d(rA(r))/dr = 1$$

$$\text{or,} \quad rA(r) = r + \text{const.} = r + m \qquad (20)$$

so that

$$A(r) = 1 + m/r\,,$$

$$B(r) = 1/A(r) = (1 + m/r)^{-1} \qquad (21)$$

here m is a constant of integration to be determined.

We can now write down the Schwarzschild line element (13) in the form

$$d\tau^2 = (1 + m/r)\, dt^2 - (1+ m/r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \qquad (22)$$

Let us emphasise here that there are ten equations of Einstein

$$R_{\mu\nu} = 0 \qquad (23)$$

and we have used only three of them to determine the form of the Schwarzschild line element. Therefore, it remains to be shown that the seven equations are consistent with the solution in (22). In fact it can be easily shown that

$$R_{\mu\nu} = 0, \qquad \text{for } \mu \neq \nu \ ,$$

$$R_{33} = \sin^2\Theta\, R_{22} = 0 \qquad (24)$$

so that all the ten equations are consistent with the line element (22).

To determine the constant o integration m, let us note that very far away from a star of mass M we have seen that the metric has the form

$$g_{00} = 1 + 2\,\varnothing(r) = 1 - 2G_N M/r \qquad (25)$$

Where M denotes the mass of the star. Comparing this with the solution in (22) we determine the constant of integration to be

$$m= - 2G_N M, \qquad (26)$$

so that the Schwarzschild line element (22) takes the final form

$$d\tau^2 = (1 - 2G_N M /r)\, dt^2 - (1- 2G_N M /r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \qquad (27)$$

This determines the form of the line element and, therefore, the metric uniquely.

One striking feature of the Schwarzschild metric (27) is that at $r = 2G_N M$,

$$g_{00} = 0, \; g_{rr} \rightarrow \infty \tag{28}$$

That is, the Schwarzschild metric is singular at the Schwarzschild radius defined by

$$r_s = 2G_N M$$

For most objects, this radius lies inside the object. For example, since

$$G_N \approx 7 \times 10^{-29} \text{ cm gm}^{-1},$$
$$M(\text{earth}) \approx 6 \times 10^{24} \text{ kg} = 6 \times 10^{27} \text{ gm}, \tag{29}$$

The Schwarzschild radius for earth has the value

$$r_s(\text{earth}) = 2G_N M(\text{earth})$$
$$\approx 0.84 \text{ cm}, \tag{30}$$

Which is well inside the earth.

Acknowledgement:

The material presented in this report has been collected from the following sources:

- Wikipedia
- INTRODUCTION TO GENERAL RELATIVITY by Gerard ’t Hooft
- Lectures on Gravitation by Ashok Das
- Mathematica Software Package tutorials.

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