Estimation of parameters of a partial neutrino mixing matrix

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Contents

Chapter 1

Introduction

The partial neutrino mixing matrix predicts the three neutrino mixing angles in terms of two parameters. These two parameters can be constrained using the experimental value of the three neutrino mixing angles. A chi-square analysis is used to extract the value of these two parameters. Moreover, a Monte Carlo study is used to check accuracy of the result.

1.1 Mixing Matrix

In 1968 during an experiment at home-stake it had been observed that the flux of electron neutrinos coming from sun was only about one third of that predicted by Behcall. The so called 'Solar neutrino problem'. Solution of this problem was purposed by B. Pontecorvo that electron neutrinos produced by sun are transformed in flight into different species, to which home-stake experiment was insensitive. Later from experiments like Super-kamiokande and SNO it was confirmed that electron neutrinos *were transforming into muon neutrinos. This is the mechanism we now call neutrino oscillation.*

In atmosphere neutrinos are produced from decay of pions and muons.

$$\pi^+ \to \mu^+ + \nu_\mu, \qquad \mu^+ \to e^+ + \nu_e + \bar{\nu_\mu}$$

$$\pi^- \to \mu^- + \bar{\nu_\mu}, \qquad \mu^- \to e^- + \bar{\nu_e} + \nu_\mu$$

Both these reactions produce muon neutrino whereas electron neutrino is produced only in one reaction. Evidently it was supposed that there should be twice as many muon neutrino as electron neutrino. Indeed this was the case for neutrinos coming directly from overhead which travell a distance of 10 Km or so. As the zenith angle increases and with this distance from the source, ratio of muon to electron neutrino decreases. This suggest muon neutrinos are also changing into tau neutrino.

Neutrinos interact as flavour eigenstate$(\nu_e, \nu_\mu, \nu_\tau)$ *and they travel as eigenstates of free particle Hamiltonian which are* $(\nu_e, \nu_\mu, \nu_\tau)$ *called mass eigenstates. The theory is similar to quantum mechanics of mixed states. flavour eigenstates are linear combination of mass eigenstates.*

$$\nu_e = U_{11}\nu_1 + U_{12}\nu_2 + U_{13}\nu_3$$

$$\nu_\mu = U_{21}\nu_1 + U_{22}\nu_2 + U_{23}\nu_3 \tag{1.1}$$

$$\nu_\tau = U_{31}\nu_1 + U_{32}\nu_2 + U_{33}\nu_3 \tag{1.2}$$

These equations can also be written in matrix form:-

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Matrix on right hand side is called Mixing matrix. General way of constructing this matrix is: -

$$U = R_{23}(\theta_{23})R_{13}(\theta_{13}, \delta)R_{12}(\theta_{12})$$

Where,

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} * e^{-\iota\delta} \\ 0 & 1 & 0 \\ -s_{13} * e^{\iota\delta} & 0 & c_{13} \end{pmatrix}$$

$$R_{12} = \begin{pmatrix} c12 & s12 & 0 \\ s12 & c12 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\iota\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{\iota\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{\iota\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{\iota\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{\iota\delta} & c_{23}c_{13} \end{pmatrix}$$

In general case a mixing matrix has four parameters, three mixing angles and a CP violating phase. However we can reduce free parameters by using flavour symmetries.

Flavour symmetries reduce free parameters either by relating or predicting some of them. We are studying the partial mixing matrix whose one column is determine by symmetries and other two colkumns are fixed by unitary conditions. e.g

$$U = \begin{pmatrix} a & * & * \\ b & * & * \\ c & * & * \end{pmatrix} \qquad \text{Where a=2, b=c=1}$$

Our U is a function of two parameters $(\theta$ and $\phi)$

1.2 Statistical Methods

1.2.1 Error Propagation

We often want to determine a dependent variable q that is a function of one or more different measured variables. Experientially, variables are often measured with uncertainties. Then how these uncertainties propagate into the dependent variable q ?

Case I

Suppose we measured a quantity x and proceed to calculate the quantity

$$q = x + A$$

Where A is a fixed number with no uncertainty. Suppose also that the measurements of x are normally distributed about true valuie X with width σx

Probability of obtaining any value x in the interval dx is :

$$e^{\frac{-(x-X)^2}{2\sigma_x^2}}$$

Our problem is to obtain the probability of any value q=x+A, or x=q-A

probability of obtaining value x = Probability of obtaining value x=q-A

$$e^{\frac{-[q-(X+A)]^2}{2\sigma_x^2}}$$

This result shows that calculated values of q are centred at X+A with uncertainty σ_x

Case II

This time consider quantity to be calculated as

$q = Ax$ *where* A *is a fixed number.*

Proceeding same as before we can say that

Probability of obtaining value q = Probability of obtaining value x=q/b

$$e^{\frac{-(q-BX)^2}{2B^2\sigma_x^2}} \tag{1.3}$$

In other words the values of q=BX will be normally distributed , with center at q=BX and uncertainty $B\sigma_x$

General Case

Suppose we want to measure two independent quantities x and y whose observed values are normally distributed, and we now calculate quantity $q(x, y)$*. We* *are making an assumption that there uncertainties σ_x and σ_y are very small as compared to central value X and Y. This assumption means that we are concerned with only that values of x and y which are very close to X and Y. Then we can write q(x,y) as*

$$q(x, y) \approx q(X, Y) + \left(\frac{\partial q}{\partial x}\right) \delta x + \left(\frac{\partial q}{\partial y}\right) \delta y$$

This approximation is good because most of the time values occur close to central values X and Y. Two partial derivatives are fixed numbers because they are evaluated at X and Y   Here first term is simply a constant term so it mainly shifts the distribution of answers. The second number is the fixed number $\frac{\partial q}{\partial x}$ times δx whose distribution has width (or uncertainty in x) σx, so the values of second term are centred on zero with width

$$\left(\frac{\partial q}{\partial x}\right) \delta x$$

similarly the values of third term are centred on zero with width

$$\left(\frac{\partial q}{\partial y}\right) \delta y$$

combining above three equations we can say that values of q(x,y) are normally distributed about the true value q(X,Y) with width

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \left(\frac{\partial q}{\partial y}\delta y\right)^2}$$

If we identify the standard deviation σ_x and σ_y as uncertainties in x and y then above result is precisely the rule for propagation of random errors for the case when q is function of just two variables. If q depends on several variables then this equation can be generalised easily as

$$\sigma_q = \sqrt{(\frac{\partial q}{\partial x}\delta x)^2 + (\frac{\partial q}{\partial y}\delta y)^2.....}$$

1.2.2 χ^2

Suppose we perform some experiment for which we know the distribution of experimental results. We perform experiment several times to record our observations, now question is how can we decide whether our observed values are in agreement with theoretical values. This can be done by χ^2 test. Consider during experiment we get values of dependent variables q_i. Theoretically q_i's are functions of two unknown variables x and y. If O_i and E_i represents observed and expected values respectively then χ^2 can be given as

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{p_i - f_{p_i}(x,y)}{dp_i^{exp}} \right)^2$$

In numerator Difference signifies that we are comparing observed and expected value of p_i. Square is used so that we always get positive value of difference because if we do not take square then difference can be negative and negative errors can cancel other errors leading to underestimate of errors. It can happen that during experiment that we measure p_1 with very large accuracy but some other variable p_2 with small accuracy. Then in χ^2 weightage of term including p_1 should be greater then that of p_2. That is why we divide the numerator by dp_i

During estimation of parameters (Here x and y) We replace E_i with expression for E_i then we plot χ^2 either contour plot or 3D plot to know minimum value of χ^2.

Minimum value of χ^2 curve corresponds to best fit values of x and y.

Monte Carlo

Practically Monte Carlo provides a mathod of simulating experiment and creating models of experimental data. With a Monte Carlo calculation, we can test the statistical significance of data with relative simple calculations that require neither a deep theoretical understanding of statistical analysis nor sophisticated programming technique.

We have three equations of mixing angles in terms of two known variables θ and ϕ. We know there experimental values and uncertainties in them. We also know the distribution followed by θ and ϕ. We will generate 10000 values of θ and ϕ randomly around there central value. These values when inserted into expressions of mixing angles we will get random values of mixing angles. By making histogram of these values of mixing angles we can know standard deviation and central values of mixing angles.

Chapter 2

Results

2.0.3 Estimation of θ and ϕ using χ^2

$$\sin^2\theta_{13} = \frac{\sin^2\theta}{3} \tag{2.1}$$

$$\sin^2\theta_{12} = 1 - \frac{a^2}{(a^2+b^2+c^2)\cos\theta_{13}} \tag{2.2}$$

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{\left(abc\sin 2\theta\sqrt{a^2+b^2+c^2}\right)}{(b^2+c^2)(a^2+(b^2+c^2)\cos^2\theta)}\cos\phi \tag{2.3}$$

Experimental Values of mixing angles are

- $\sin^2\theta_{13} = 0.026 \pm 0.003$
- $\sin^2\theta_{12} = 0.32 \pm 0.01$
- $\sin^2\theta_{23} = 0.49 \pm 0.08$

and χ^2 *expression is:-*

$$\chi^2 = \left(\frac{0.026 - \sin^2\theta_{13}}{0.003}\right)^2 + \left(\frac{0.49 - \sin^2\theta_{23}}{0.08}\right)^2 + \left(\frac{0.32 - \sin^2\theta_{12}}{0.01}\right)^2$$

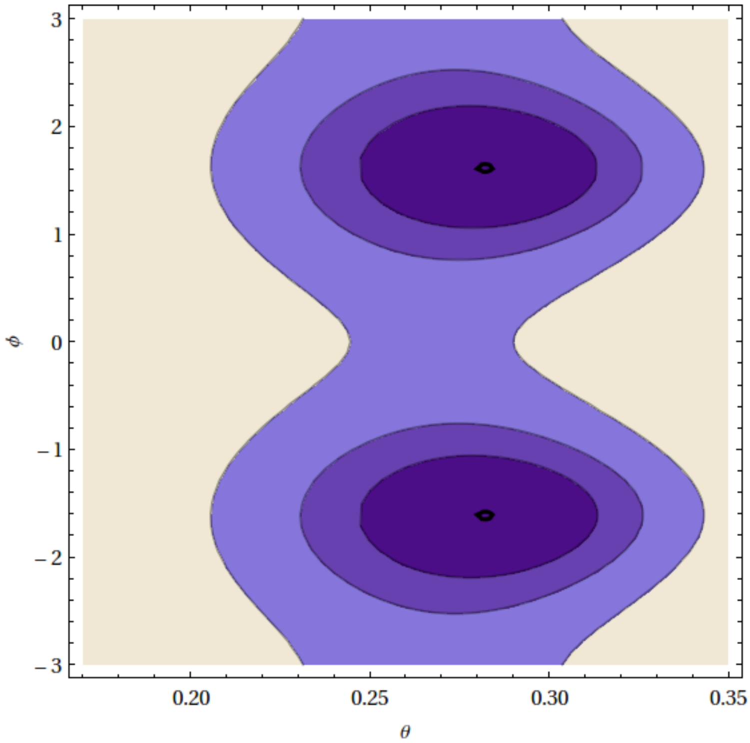


Figure 2.1: Here black region represent the minimized value of θ and ϕ. Region outside this is contour corresponding to 1σ. Minimum numerical value of χ^2 is 0.07 corresponding to $\theta = 0.28 \pm 0.03$ and $\phi = 1.61 \pm 0.6$. Width and height of 1σ contour gives us uncertainties for central values of θ and ϕ

Mathematica Code - χ^2

```
a = Sqrt[2/3];
b = 1/Sqrt[6];
c = 1/Sqrt[6];
ts13[θ_] := Sin[θ]^2/3
chis13[θ_] := ((0.026 - ts13[θ])/0.004)^2 (*Chi Square Value of Sin^2θ13*)
(*Calculation of Chi Square for Sin^2θ23*)
tA[θ_] := (a * b * c * Sin[2 θ] Sqrt[a^2 + b^2 + c^2])/((b^2 + c^2) (a^2 + (b^2 + c^2) Cos[θ]^2))
ts23[θ_, ϕ_] := 0.5 + tA[θ] * Cos[ϕ]

chis23[θ_, ϕ_] := ((ts23[θ, ϕ] - 0.49)/0.08)^2 (*Chi Square for Sin^2θ23*)
(*Calculation of Chi Square for Sin^2θ12*)
tθ13[θ_] := ArcSin[Sqrt[ts13[θ]]]
tθ12[θ_] := ArcCos[(a * Sec[tθ13[θ]])/Sqrt[a^2 + b^2 + c^2]]
tsin12[θ_] := Sin[tθ12[θ]]^2
chis12[θ_] := ((0.32 - tsin12[θ])/0.017)^2
(*Final Chi Squared*)
chisquare[θ_, ϕ_] := chis13[θ] + chis23[θ, ϕ] + chis12[θ]

Plot3D[chisquare[θ, ϕ], {θ, 0.17, 0.35}, {ϕ, -3, 3}]
kk1 = ContourPlot[chisquare[θ, ϕ], {θ, 0.17, 0.35},
   {ϕ, -3, 3}, Contours → {2.30, 4.61, 9.21}, FrameLabel → Automatic];
FindMinimum[chisquare[θ, ϕ], {{θ, 0.2}, {ϕ, 1.5}}](*Minimum Value of θ and ϕ*)
kk2 = ContourPlot[chisquare[θ, ϕ] == 0.0779592,
   {θ, 0.17, 0.35}, {ϕ, -3, 3}, ContourStyle → Directive[Black, Thick]];
Show[
 kk1,
 kk2]
```

2.1 Error Propagation

We can calculate errors in mixing angles by using expressions previously discussed in 1.2.1 to get

$$\sigma_{13} = \frac{\sin 2\theta}{3}\delta\theta$$

$$\sigma_{12} = \quad 2(Na)^2 \sec^2\theta_{13}\tan\theta_{13}\delta\theta_{13} \tag{2.4}$$

$$\textit{where } \delta\theta_{13} = \quad \frac{\cos\theta}{\sqrt{3}\sqrt{1-\frac{\sin^2\theta}{3}}}\delta\theta \tag{2.5}$$

$$\textit{and } N = \quad \frac{1}{\sqrt{a^2+b^2+c^2}} \tag{2.6}$$

$$\sigma_{23} = \quad \sqrt{(\frac{\delta A}{A})^2 + (\frac{\delta\cos\phi}{\cos\phi})^2} \tag{2.7}$$

$$\textit{where } \delta A = \quad \left(\frac{\sqrt{\frac{2}{3}}Cos[2\theta]}{\frac{2}{3}+\frac{Cos[\theta]^2}{3}} + \frac{\sqrt{\frac{2}{3}}Cos[\theta]Sin[\theta]Sin[2\theta]}{3\left(\frac{2}{3}+\frac{Cos[\theta]^2}{3}\right)^2}\right)\delta\theta \tag{2.8}$$

$$\textit{and } \delta\cos\phi = \quad \sin\phi\delta\phi \tag{2.9}$$

Experimental values of mixing angles:-

- $\sin^2\theta_{13} = 0.026 \pm 0.004$
- $\sin^2\theta_{12} = 0.32 \pm 0.01$
- $\sin^2\theta_{23} = 0.49 \pm 0.08$

Values of mixing angles from error propagation: -

- $\sin^2\theta_{13} = 0.026 \pm 0.005$
- $\sin^2\theta_{12} = 0.32 \pm 0.01$

- $\sin^2\theta_{23} = 0.49 \pm 16.37$

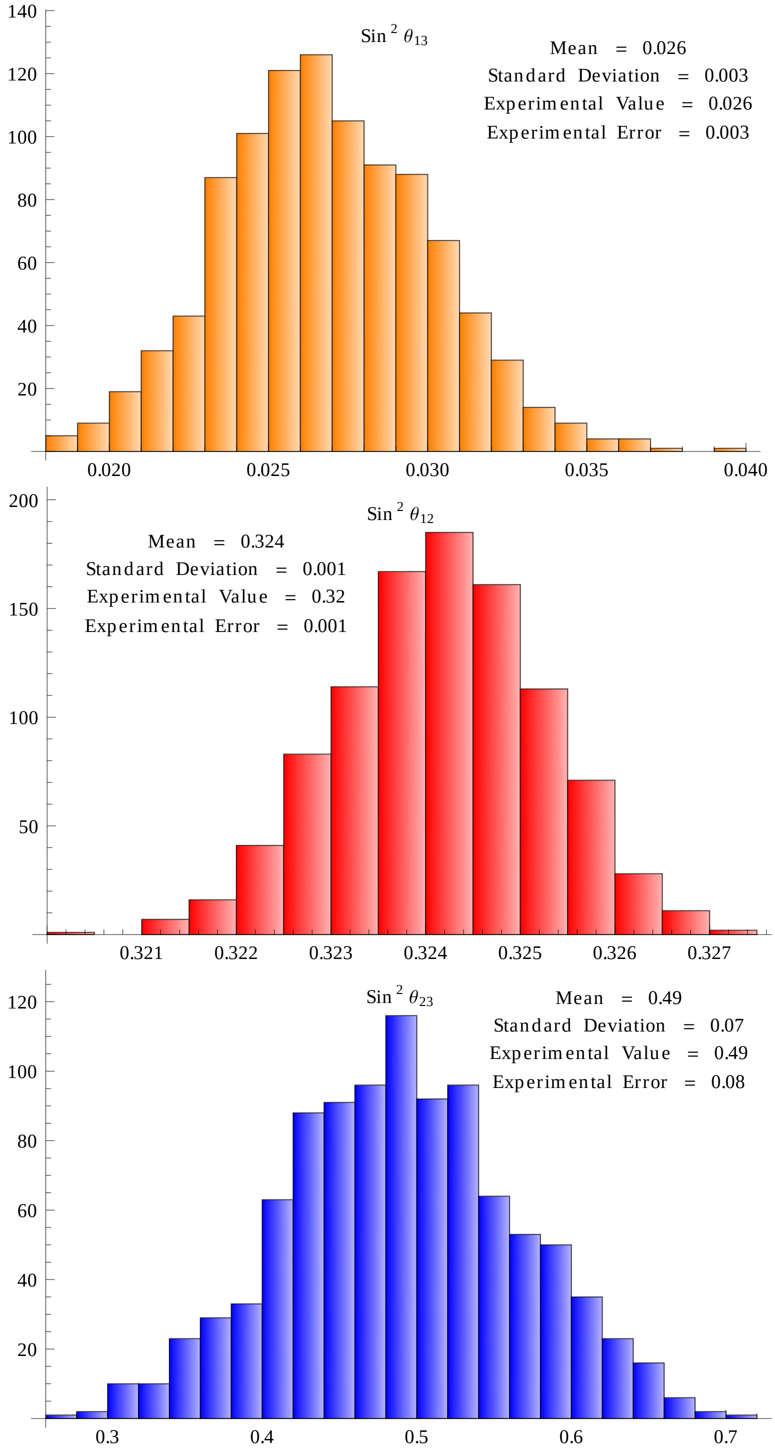
Here error in $\sin^2\theta_{23}$ *is very large because in expression of error propagation for* $\sin^2\theta_{23}$ *there is* $\cos\phi$ *term in denominator which goes to* $\inf$ *as* ϕ *goes to 90 degree.*

Mathematica Code - Error Propagation

```
(*************************************************************************************
                  *************************************)
(*Error Propagation in Mathematica 8*)
Quit[]
a = Sqrt[2/3];
b = 1/Sqrt[6];
c = 1/Sqrt[6];
θ = 0.2838;
δθ = 0.03;
ϕ = -1.61;
δϕ = 1.3;
Print["True Value of Sinθ^2_13 is : - "]
ts13 = Sin[θ]^2/3 (*True Value Of Sinθ^2_13*)
Print["Error in the value of Sinθ^2_13 is : - "]
es13 = Sin[2 θ] * δθ
(*fes13=(2*Cos[θ])/(3*Sin[θ])*δθ  ;   (*Error in Sin^2θ_13*)
es13= fes13*ts13;*)

tA = (a * b * c * Sin[2 θ] Sqrt[a^2 + b^2 + c^2])/((b^2 + c^2) (a^2 + (b^2 + c^2) Cos[θ]^2));(*True Value of A*)
Print["True Value of Sinθ^2_23 is : - "]
ts23 = c^2/(b^2 + c^2) + tA * Cos[ϕ](*True Value of Sinθ_23*)
ecϕ = Sin[ϕ] * δϕ;
numeratortA = a * b * c * Sin[2 θ] Sqrt[a^2 + b^2 + c^2] ;
denominatortA = (b^2 + c^2) (a^2 + (b^2 + c^2) Cos[θ]^2);
numeratoreA = a * b * c * 2 * 2 * Cos[2 θ] * Sqrt[a^2 + b^2 + c^2] * δθ;
denominatoreA = 2 * Sin[θ] * δθ (b^2 + c^2)^2;
fraceA = numeratoreA/numeratortA + denominatoreA/denominatortA ;
eA = fraceA * tA;
```

2.2 Monte Carlo



Here in all the histograms we get same best fit and uncertainties of mixing angles as given by experiments which verifies our result of χ^2

Mathematica Code - Monte Carlo

```
a = Sqrt[2/3];
b = 1/Sqrt[6];
c = 1/Sqrt[6];

ttheta := RandomVariate[NormalDistribution[16.5 °, 1 °]]
pphi := RandomVariate[NormalDistribution[-91.85 °, 20 °]]
ss13 := Sin[ttheta]^2/3;
(*Sin13^2*)
ttheta13 := ArcSin[Sqrt[ss13]];(*θ13*)
ss12 := 1 - a^2/((a^2 + b^2 + c^2) Cos[ttheta13]);(*Sin12^2*)

ss23 := 0.5 + (a*b*c*Sin[2*ttheta] Sqrt[a^2 + b^2 + c^2])/((b^2 + c^2) (a^2 + (b^2 + c^2) Cos[ttheta]^2)) * Cos[pphi]

Histogram[Labeled[Table[ss13, {j, 1, 1000}], "Sin^2θ13", Above],
 ChartElementFunction → "FadingRectangle", ChartStyle → Orange]
Histogram[Labeled[Table[ss12, {j, 1, 1000}], "Sin^2θ12", Above],
 ChartElementFunction → "FadingRectangle", ChartStyle → Red]
Histogram[Labeled[Table[ss23, {j, 1, 1000}], "Sin^2θ23", Above],
 ChartElementFunction → "FadingRectangle", ChartStyle → Blue]
n = 10000;
Print["Mean Value of Sin^2θ12 "]
mss12 = Mean[Table[ss12, {j, 1, n}]]
Print["Mean Value of Sin^2θ23 "]
mss23 = Mean[Table[ss23, {j, 1, n}]]
Print["Mean Value of Sin^2θ13 "]
mss13 = Mean[Table[ss13, {j, 1, n}]]
Print["Standard Deviation of Sin^2θ12"]
StandardDeviation[Table[ss12, {j, 1, n}]]
Print["Standard Deviation of Sin^2θ23"]
StandardDeviation[Table[ss23, {j, 1, n}]]
```

Chapter 3

Summary

We studied how to estimate parameters using χ^2 *which we then used to estimate two variables from experimental values of three mixing angles. These results are verified first using error propagation and then Monte Carlo method. Uncertainties for* $\sin^2\theta_{12}$ *given by error propagation are not matching with experimental values because error propagation assumes that uncertainty should be very small as compared to central value but in case of* $\sin^2\theta_{12}$ *uncertainty becomes infinite as* ϕ *approaches 90 degree. However results provided by Monte Carlo are in exact match with experimental values verifying our result of* χ^2.