A

PROJECT

REPORT

On

Numerical solutions of ordinary differential equation using runge kutta method

Submitted by:

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Numerical Solution of Ordinary Differential Equations (ODE)

I. Definition

An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations. So, it is important for engineers and scientists to know how to set up differential equations and solve them.

Differential equations are of two types

1) ordinary differential equation (ODE)
2) partial differential equations (PDE).

An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable. Examples of ordinary differential equation include

1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0, \frac{dy}{dx}(0) = 2, y(0) = 4,$
2) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = \sin x, \frac{d^2y}{dx^2}(0) = 12, \frac{dy}{dx}(0) = 2, y(0) = 4$

Note: In this part, we will see how to solve ODE of the form

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

II. Euler's Method

We will use Euler's method to solve an ODE under the form:

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

At $x = 0$, we are given the value of $y = y_0$. Let us call $x = 0$ as x_0. Now since we know the slope of y with respect to x, that is, $f(x, y)$, then at $x = x_0$, the slope is $f(x_0, y_0)$. Both x_0 and y_0 are known from the initial condition $y(x_0) = y_0$.

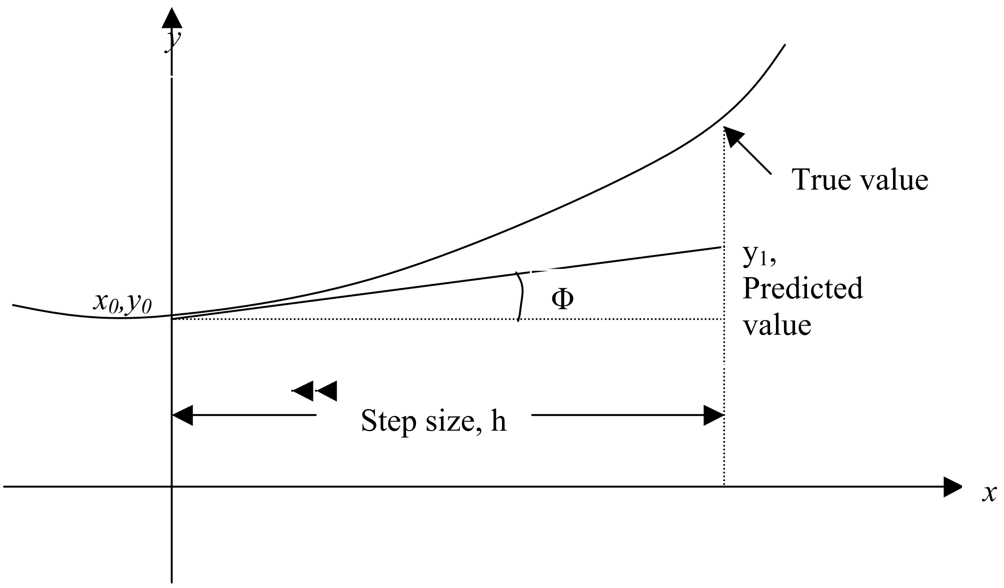

Figure 1. Graphical interpretation of the first step of Euler's method.

So the slope at x=$x_0$as shown in the figure above

$$\text{Slope } = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= f(x_0, y_0)$$

Thus

$$y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$$

If we consider $x_1 - x_0$ as a step size h, we get

$$y_1 = y_0 + f(x_0, y_0)\,h.$$

We are able now to use the value of y_1 (an approximate value of y at $x = x_1$) to calculate y_2, which is the predicted value at x_2,

$$y_2 = y_1 + f(x_1, y_1)\,h$$
$$x_2 = x_1 + h$$

Based on the above equations, if we now know the value of $y = y_i$ at x_i, then

$$\boxed{y_{i+1} = y_i + f(x_i, y_i)\,h}$$

This formula is known as the Euler's method and is illustrated graphically in Figure 2. In some books, it is also called the Euler-Cauchy method.

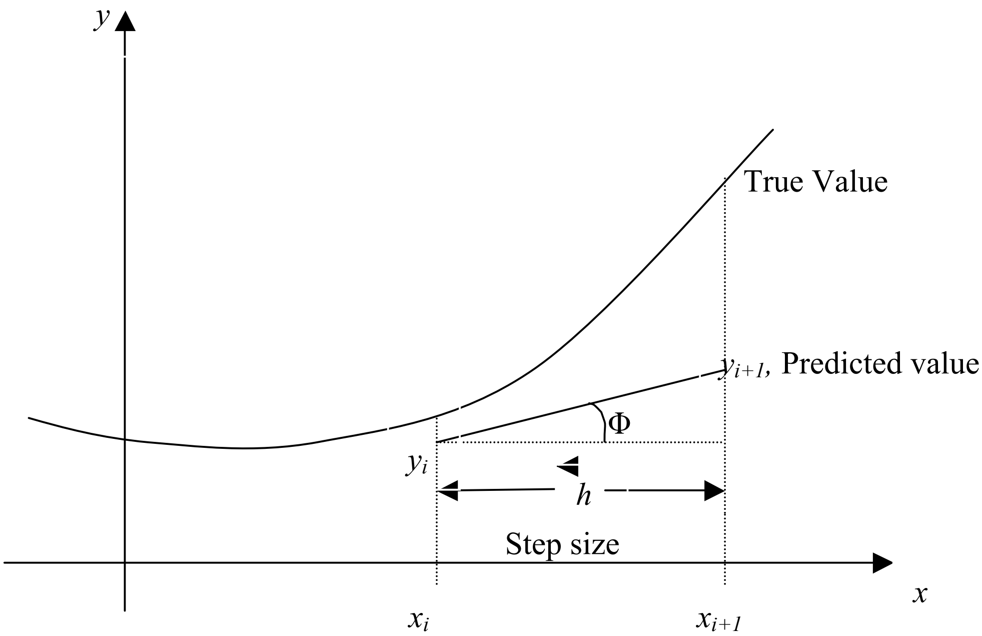

Figure 2. General graphical interpretation of Euler's method.

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \ldots$$

The true error hence is approximately proportional to the square of the step size, that is, as the step size is halved, the true error gets approximately quartered. However from Table 1, we see that as the step size gets halved, the true error only gets approximately halved. This is because the true error being proportioned to the square of the step size is the local truncation error, that is, error from one point to the next. The global truncation error is however proportional only to the step size as the error keeps propagating from one point to another.

II. Runge-Kutta 2nd order

Euler's method was derived from Taylor series as:

$$y_{i+1} = y_i + f(x_i, y_i)h$$

This can be considered to be Runge-Kutta 1st order method.
The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \ldots$$

Now let us consider a 2nd order method formula. This new formula would include one more term of the Taylor series as follows:

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

Let us now apply this to a simple example:

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5$$

$$f(x, y) = e^{-2x} - 3y$$

Now since *y* is a function of *x*,

$$\begin{aligned} f'(x, y) &= \frac{\partial f(x, y)}{\partial x} + \frac{\partial f(x, y)}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial}{\partial x}\left(e^{-2x} - 3y\right) + \frac{\partial}{\partial y}\left[\left(e^{-2x} - 3y\right)\right]\left(e^{-2x} - 3y\right) \\ &= -2e^{-2x} + (-3)\left(e^{-2x} - 3y\right) \\ &= -5e^{-2x} + 9y \end{aligned}$$

The 2nd order formula would be

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$$

$$y_{i+1} = y_i + \left(e^{-2x_i} - 3y_i\right)h + \frac{1}{2!}\left(-5e^{-2x_i} + 9y_i\right)h^2$$

You could easily notice the difficulty of having to find $f'(x, y)$ in the above method. What Runge and Kutta did was write the 2nd order method as

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

This form allows us to take advantage of the 2nd order method without having to calculate $f'(x, y)$.
But, how do we find the unknowns a_1, a_2, p_1 and q_{11}? Equating the above equations:

$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2$ and $y_{i+1} = y_i + (a_1k_1 + a_2k_2)h$

gives three equations.

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

Since we have 3 equations and 4 unknowns, we can assume the value of one of the unknowns. The other three will then be determined from the three equations. Generally the value of a_2 is chosen to evaluate the other three constants. The three values generally used for a_2 are $\frac{1}{2}$, 1 and $\frac{2}{3}$, and are known as Heun's Method, Midpoint method and Ralston's method, respectively.

II.1. Heun's method

Here we choose $a_2 = \frac{1}{2}$, giving

$$a_1 = \frac{1}{2}$$
$$p_1 = 1$$
$$q_{11} = 1$$

resulting in

$$\boxed{y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h}$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

This method is graphically explained in Figure 6.

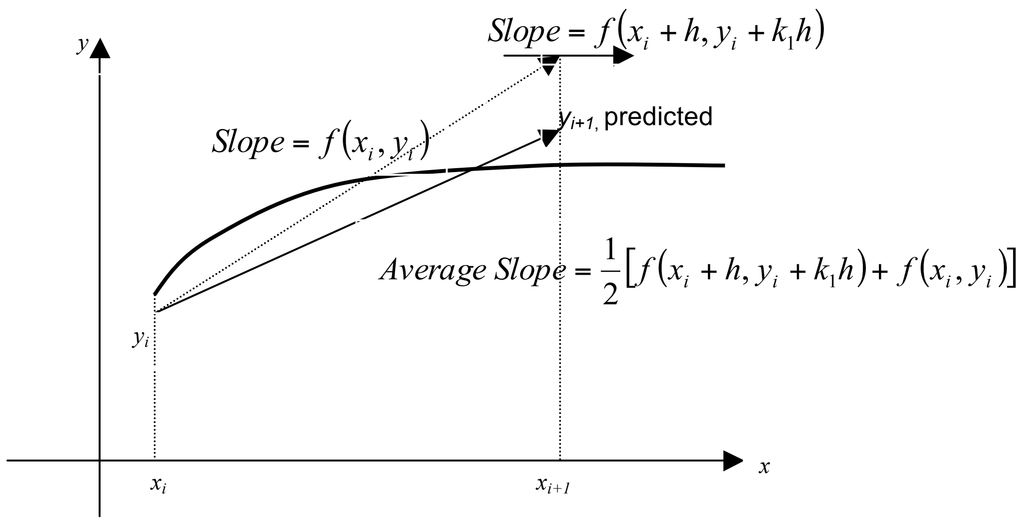

Figure 6. Runge-Kutta 2nd order method(Heun's method).

II.2. Midpoint method

Here we choose $a_2 = 1$, giving

$$a_1 = 0$$
$$p_1 = \frac{1}{2}$$
$$q_{11} = \frac{1}{2}$$

resulting in

$$\boxed{y_{i+1} = y_i + k_2 h}$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

II.3. Ralston's method

Here we choose $a_2 = \frac{2}{3}$, giving

$$a_1 = \frac{1}{3}$$
$$p_1 = \frac{3}{4}$$
$$q_{11} = \frac{3}{4}$$

resulting in

$$\boxed{y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h}$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

NOTE: *How do these three methods compare with results obtained if we found $f'(x, y)$ directly?*

We know that since we are including first three terms in the series, if the solution is a polynomial of order two or less (that is, quadratic, linear or constant), any of the three methods are exact. But for any other case the results will be different.

Consider the following example

$$\frac{dy}{dx} = e^{-2x} - 3y, y(0) = 5$$

If we directly find the $f'(x, y)$, the first three terms of Taylor series gives

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f'(x_i, y_i)h^2$$

where

$$f(x, y) = e^{-2x} - 3y$$
$$f'(x, y) = -5e^{-2x} + 9y$$

For a step size of $h = 0.2$, using Heun's method, we find

$$y(0.6) = 1.0930$$

The exact solution

$$y(x) = e^{-2x} + 4e^{-3x}$$

gives

$$y(0.6) = e^{-2(0.6)} + 4e^{-3(0.6)}$$
$$= 0.96239$$

Then the absolute relative true error is

$$|\in_t| = \left|\frac{0.96239 - 1.0930}{0.96239}\right| \times 100$$
$$= 13.571\%$$

For the same problem, the results from the Euler and the three Runge-Kuttamethod are given below

Comparison of Euler's and Runge-Kutta 2nd order methods

	y(0.6)					
	Exact	Euler	Direct 2nd	Heun	Midpoint	Ralston
Value	0.96239	0.4955	1.0930	1.1012	1.0974	1.0994
$\|\in_t\|$%		48.514	13.571	14.423	14.029	14.236

III. Runge-Kutta 4th order

Runge-Kutta 4th order method is based on the following

$$y_{i+1} = y_i + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)h$$

where knowing the value of $y = y_i$ at x_i, we can find the value of $y=y_{i+1}$ at x_{i+1}, and

$$h = x_{i+1} - x_i$$

The above equation is equated to the first five terms of Taylor series

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i,y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i,y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i,y_i}(x_{i+1} - x_i)^3$$
$$+ \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i,y_i}(x_{i+1} - x_i)^4$$

Knowing that $\frac{dy}{dx} = f(x,y)$ and $x_{i+1} - x_i = h$

$$y_{i+1} = y_i + f(x_i,y_i)h + \frac{1}{2!}f'(x_i,y_i)h^2 + \frac{1}{3!}f''(x_i,y_i)h^3 + \frac{1}{4!}f'''(x_i,y_i)h^4$$

Based on equating the above equations, one of the popular solutions used is

$$\boxed{y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h}$$

$k_1 = f(x_i, y_i)$ This is the slope at x_i.

$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$ This is an estimate of the slope at the midpoint of the interval $[x_i, x_{i+1}]$ using the Euler method to predict the *y* approximation there.

$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2h\right)$ This is an Improved Euler approximation for the slope at the midpoint.

$k_4 = f(x_i + h, y_i + k_3h)$ This is the Euler method slope at x_{i+1}, using the Improved Euler slope k_3 at the midpoint to step to x_{i+1}.

Errors

There are two main sources of the **total error** in numerical approximations:

1. The **global truncation error** arises from the cumulative effect of two causes:

At each step we use an approximate formula to determine y_{n+1} (leading to a **local truncation error**).

The input data at each step are only approximately correct since in general (t_n) y_n.

2. **Round-off error**, also cumulative, arises from using only a finite number of digits.

It can be shown that the global truncation error for the Euler method is proportional to h, for the Improved Euler method is proportional to h^2, and for the RungeKutta method is proportional to h^4.