COURSE Phy (601)

Presentation

AREA UNDER CURVE

Submited to:

Dr.Poonam Mehta

Submited by:

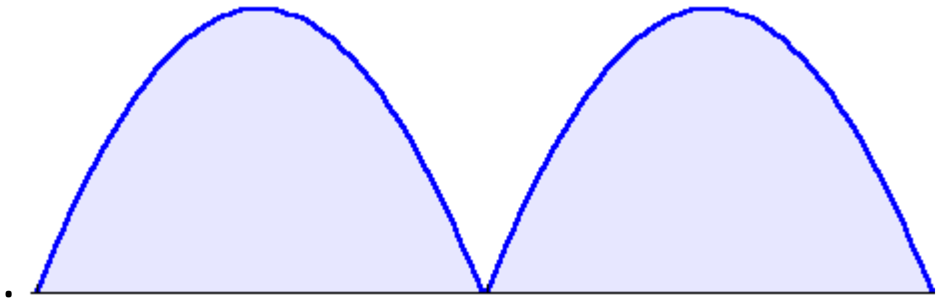
Naveen Kumar

DEPARTMENT OF PHYSICS AND ASTROPHYSICS

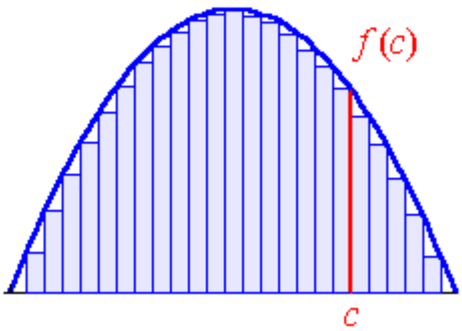
UNIVERSITY OF DELHI,DELHI-110007

Area under a curve:-

Very often we come across the process of calculating the area under some curve. We employ different methods to compute that area. Let we have a building which has parabolic archways and we want to calculate the area of building



For that what we do ,we divide the area under each parabola in large number of sub areas and compute the area of each individually,and finally add all these areas.



Here we divide the area in a large number of rectangles.

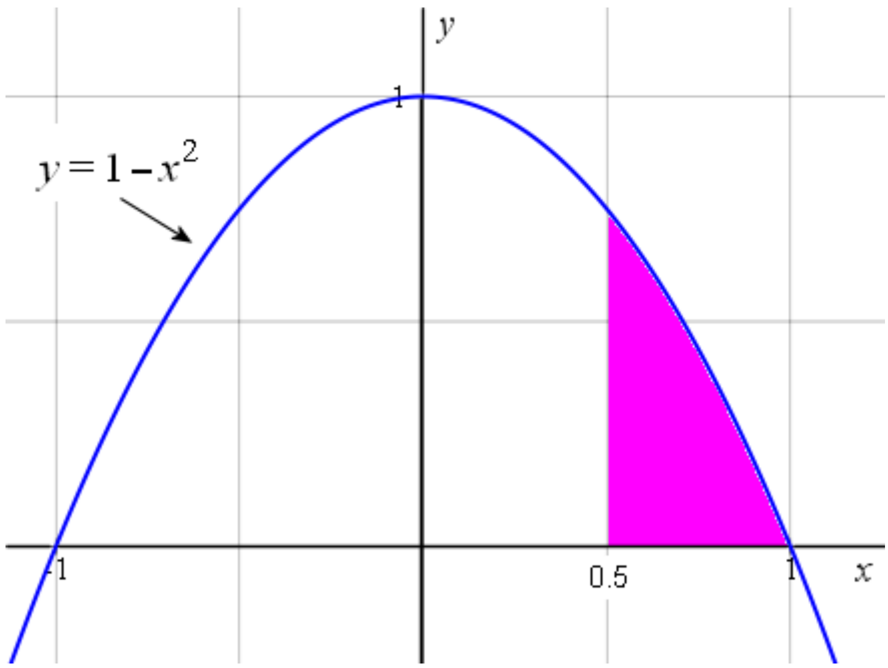
Basically I will discuss two methods to calculating the area under a curve.

1)by approximation method(area of different rectangles or triangles)

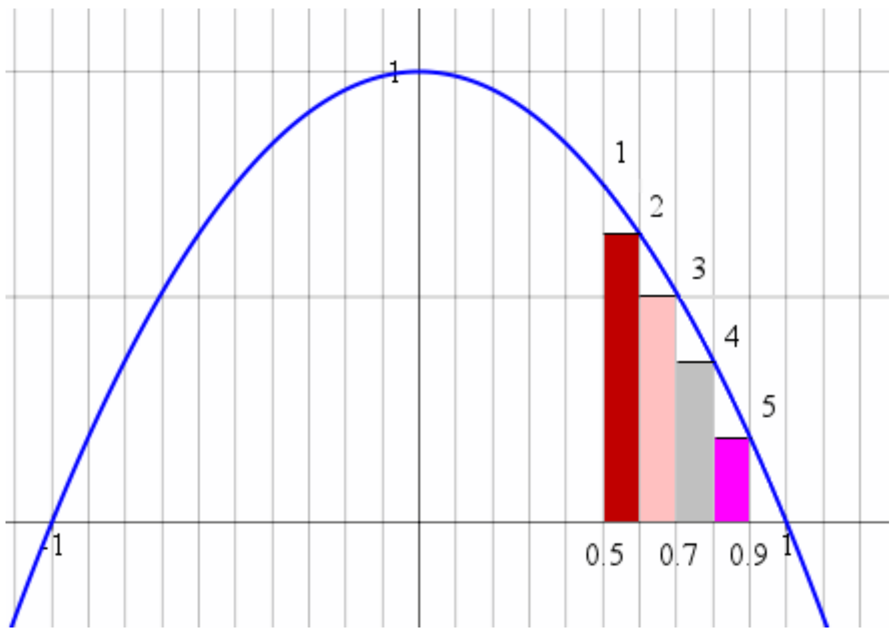
2)by integration methods

Now in approximation methods as explained earlier we divide the area in a large no of triangles or ractangles .find the area of each triangles or ractangles by some method and finally add all these areas.

Lets take an example.we want to calculate the area under the curve $y=1-x^2$ between x=.5 to x=1.0 . Here we divide the area into ractangles of equal width. Width of each rectangle is 0.1.



We want to calculate the shaded area .what we do we divide the area in a large no of rectangles.



Here we have five rectangles width of each rectangle is 0.1 and height of each rectangle is the value of function $y=1-x^2$ at that value of x.

Area of first rectangle is a=width *height.

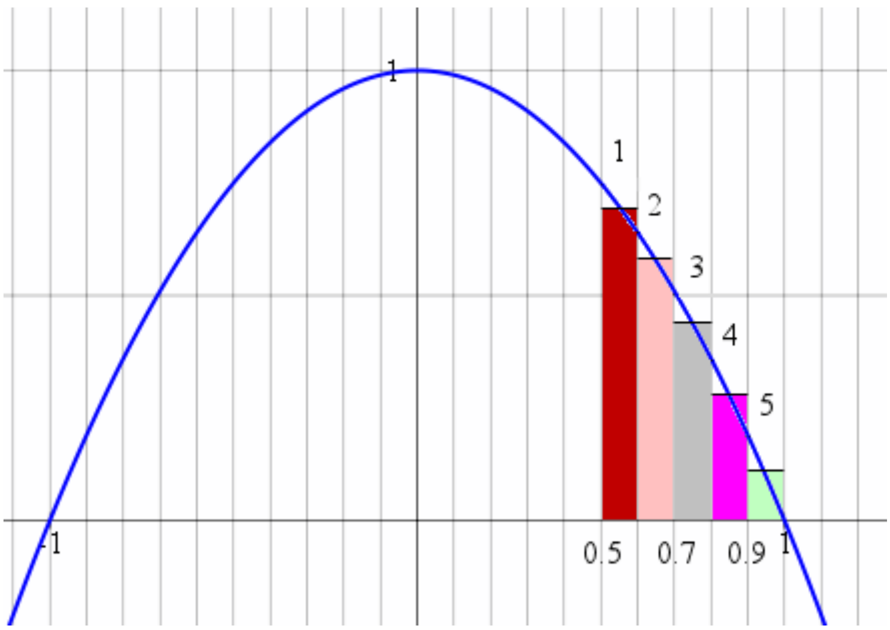
Height=$1-(0.5)^2$=0.75

A=(0.1)*(o.75)=0.075

Similarly we compute the area of rest of rectangles and add all these areas.

$$
\begin{aligned}
A &= \sum_{i=1}^{5} A_i \\
&= (0.75 + 0.64 + 0.51 + 0.36 + 0.19)(0.1) \\
&= 2.45\,(0.1) \\
&= 0.245
\end{aligned}
$$

Here we have used inner rectangles to find the areas. We can also use outer rectangles as well as mid-point rectangles. Here is the process of mid-point rectangles



Here again width of each rectangle is 0.1 and height of each rectangle is the value of function $y=1-x^2$ at that value of x.

Area of first rectangle is a=width *height.

$Height=1-(0.55)^2=0.697$

A=(0.1)*(o.697)=0.0695

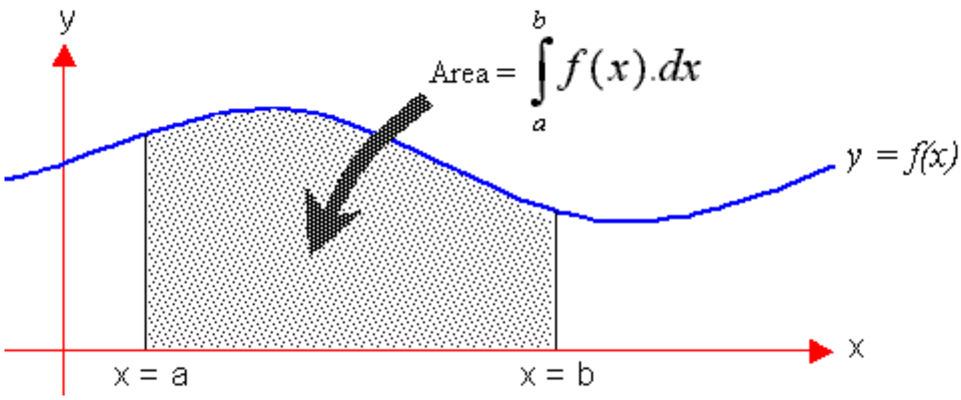
Similarly we compute the area of rest of rectangles and add all these areas.

$A= \sum_{I=1}^{5}(0.1) * (.697 + .577 + .430 + .278 + .0928) = 0.2074$

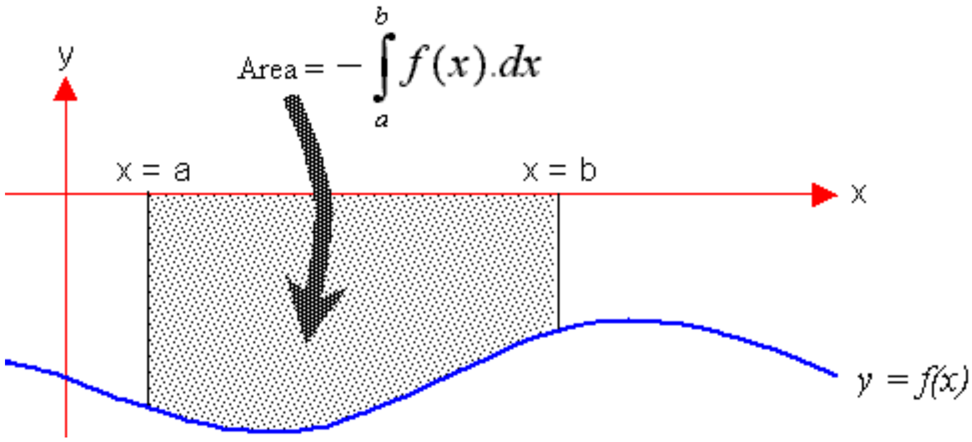
Limitation of this method is that it is quite time consuming and laborious method.so we can use second method called integration method.

The definite integral can be used to find the area between a graph curve and the 'x' axis, between two given 'x' values. This area is called the 'area under the curve' regardless of whether it is above or below the 'x' axis.

When the curve is **above** the 'x' axis, the area is the same as the definite integral ...

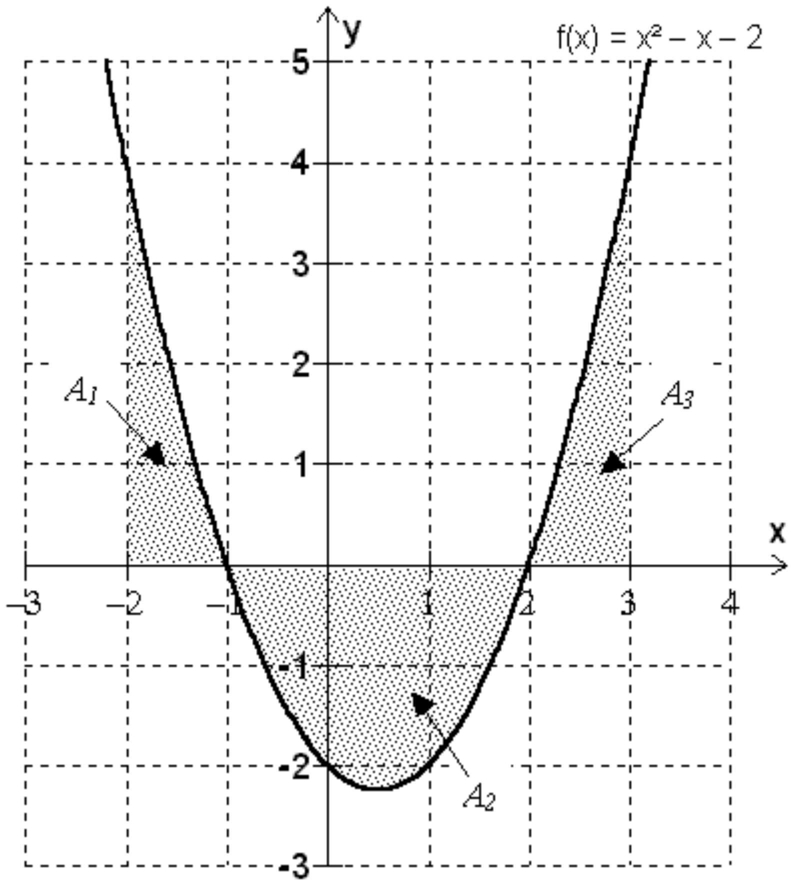


but when the graph line is **below** the 'x' axis, the definite integral is **negative**. The area is then given by:



Sometimes part of the graph is above the ‘x’ axis and part is below, then it is necessary to calculate several integrals. When the area of each part is found, the total area can be found by adding the parts.

For example, to find the area between the graph of: $y = x^2 - x - 2$ and the ‘x’ axis, from $x = -2$ to $x = 3$, we need to calculate three separate integrals:



The zeros of the function f(x) that lie between -2 and 3 form the boundaries of the separate area segments.

In this case there are zeros at x = -1 and x = 2, (see graph above) and so three separate areas must be found: A1, A2 and A3 as follows:

$$A_1 = \int_{-2}^{-1} \left(x^2 - x - 2\right) dx$$

$$A_2 = -\int_{-1}^{2} \left(x^2 - x - 2\right) dx$$

$$A_3 = \int_{2}^{3} \left(x^2 - x - 2\right) dx$$

So the total shaded area between the function and the graph from x = -2 to x = 3 is given by:

$A = A_1 + A_2 + A_3$

There are some standard methods to calculate the area under a curve.

1) Trapezoidal rule
2) Simpsons rule
3) Woole rule
4) Widdle rule

I will discuss trapezoidal and Simpsons rule .

The Trapezoidal Rule

For definite integrals such as

$$\int_0^1 \sqrt{1-x^3}\,dx \quad \text{or} \quad \int_0^1 e^{-x^2}\,dx$$

we can't use the Fundamental Theorem of Calculus to evaluate them since there are no elementary functions that are antiderivatives of $\sqrt{1-x^3}$ or e^{-x^2}. The best we can do is to use approximation methods for such integrals.

The trapezoidal rule is a numerical method that approximates the value of a definite integral. We consider the definite integral

$$\int_a^b f(x)\,dx.$$

We assume that $f(x)$ is continuous on $[a,b]$ and we divide $[a,b]$ into n subintervals of equal length

$$\Delta x = \frac{b-a}{n}$$

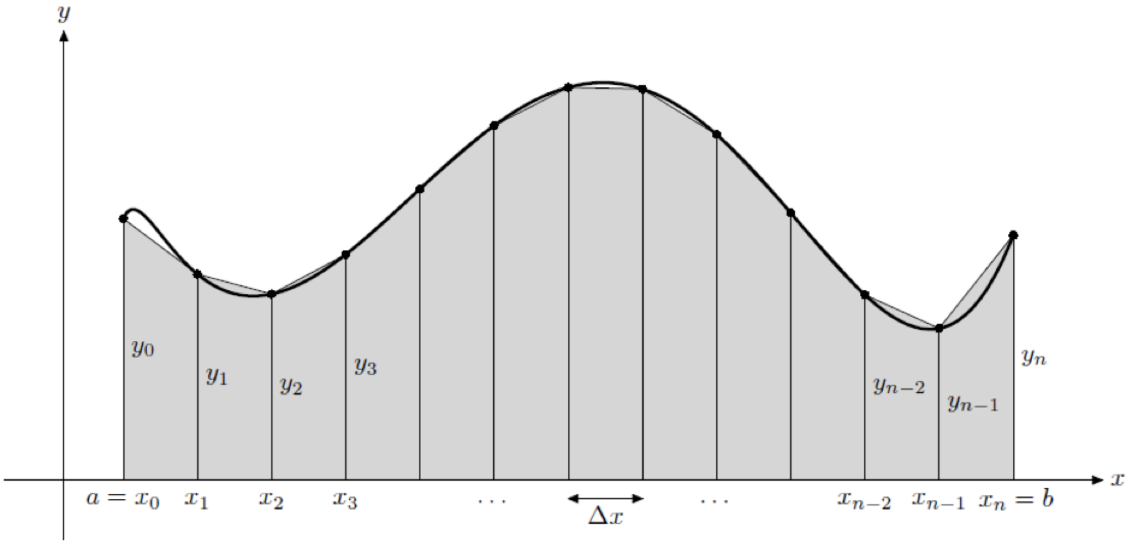
using the $n+1$ points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_n = a + n\Delta x = b.$$

We can compute the value of $f(x)$ at these points.

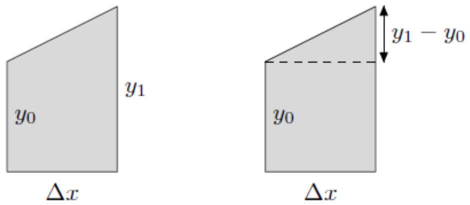
$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \ldots, \quad y_n = f(x_n)$$

We approximate the integral by using n trapezoids formed by using straight line segments between the points (x_{i-1}, y_{i-1}) and (x_i, y_i) for $1 \le i \le n$ as shown in the figure below.



The area of a trapezoid is obtained by adding the area of a rectangle and a triangle.

$$A = y_0\,\Delta x + \frac{1}{2}(y_1 - y_0)\Delta x = \frac{(y_0 + y_1)\Delta x}{2}.$$



By adding the area of the n trapezoids, we obtain the approximation

$$\int_a^b f(x)\,dx \approx \frac{(y_0 + y_1)\Delta x}{2} + \frac{(y_1 + y_2)\Delta x}{2} + \frac{(y_2 + y_3)\Delta x}{2} + \cdots + \frac{(y_{n-1} + y_n)\Delta x}{2}$$

which simplifies to the trapezoidal rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)}$$

Example 1. Use the trapezoidal rule with $n = 8$ to estimate

$$\int_1^5 \sqrt{1 + x^2}\,dx.$$

Solution. For $n = 8$, we have $\Delta x = \frac{5-1}{8} = 0.5$. We compute the values of $y_0,\ y_1,\ y_2, \ldots,\ y_8$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \sqrt{1+x^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

Therefore,

$$\begin{aligned}\int_1^5 \sqrt{1+x^2}\,dx &\approx \frac{0.5}{2}\left(\sqrt{2} + 2\sqrt{3.25} + 2\sqrt{5} + 2\sqrt{7.25} + 2\sqrt{10} + 2\sqrt{13.25} + 2\sqrt{17} + 2\sqrt{21.25} + \sqrt{26}\right)\\ &\approx \boxed{12.76}\end{aligned}$$

Example 2. The following points were found empirically.

x	2.1	2.4	2.7	3.0	3.3	3.6
y	3.2	2.7	2.9	3.5	4.1	5.2

Use the trapezoidal rule to estimate $\displaystyle\int_{2.1}^{3.6} y\,dx$.

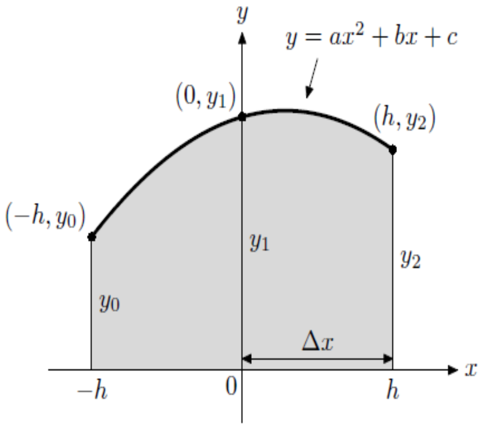
Solution. By inspection, we see that $\Delta x = 0.3$. Therefore,

$$\begin{aligned}\int_{2.1}^{3.6} y\,dx &\approx \frac{0.3}{2}(3.2 + 2(2.7) + 2(2.9) + 2(3.5) + 2(4.1) + 5.2)\\ &\approx \boxed{5.22}\end{aligned}$$

Simpson's Rule

Simpson's rule is a numerical method that approximates the value of a definite integral by using quadratic polynomials.

Let's first derive a formula for the area under a parabola of equation $y = ax^2 + bx + c$ passing through the three points: $(-h, y_0)$, $(0, y_1)$, (h, y_2).



$$
\begin{aligned}
A &= \int_{-h}^{h} (ax^2 + bx + c)\, dx \\
&= \left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx \right) \Bigg|_{-h}^{h} \\
&= \frac{2ah^3}{3} + 2ch \\
&= \frac{h}{3} \left(2ah^2 + 6c \right)
\end{aligned}
$$

Since the points $(-h, y_0)$, $(0, y_1)$, (h, y_2) are on the parabola, they satisfy $y = ax^2 + bx + c$. Therefore,

$$
\begin{aligned}
y_0 &= ah^2 - bh + c \\
y_1 &= c \\
y_2 &= ah^2 + bh + c
\end{aligned}
$$

Observe that

$$y_0 + 4y_1 + y_2 = (ah^2 - bh + c) + 4c + (ah^2 + bh + c) = 2ah^2 + 6c.$$

Therefore, the area under the parabola is

$$A = \frac{h}{3} \left(y_0 + 4y_1 + y_2 \right) = \frac{\Delta x}{3} \left(y_0 + 4y_1 + y_2 \right).$$

We consider the definite integral

$$\int_a^b f(x)\,dx.$$

We assume that $f(x)$ is continuous on $[a,b]$ and we divide $[a,b]$ into an **even** number n of subintervals of equal length

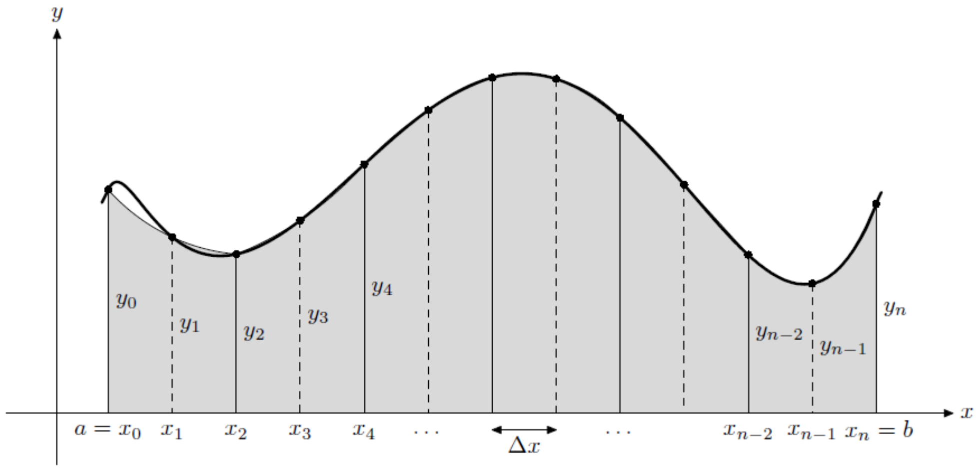
$$\Delta x = \frac{b-a}{n}$$

using the $n+1$ points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \dots, \quad x_n = a + n\Delta x = b.$$

We can compute the value of $f(x)$ at these points.

$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \dots, \quad y_n = f(x_n).$$



We can estimate the integral by adding the areas under the parabolic arcs through three successive points.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + y_2) + \frac{\Delta x}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

By simplifying, we obtain Simpson's rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)}$$

Example. Use Simpson's rule with $n = 6$ to estimate

$$\int_1^4 \sqrt{1+x^3}\,dx.$$

Solution. For $n = 6$, we have $\Delta x = \frac{4-1}{6} = 0.5$. We compute the values of $y_0, y_1, y_2, \dots, y_6$.

x	1	1.5	2	2.5	3	3.5	4
$y = \sqrt{1+x^3}$	$\sqrt{2}$	$\sqrt{4.375}$	3	$\sqrt{16.625}$	$\sqrt{28}$	$\sqrt{43.875}$	$\sqrt{65}$

Therefore,

$$\int_1^4 \sqrt{1+x^3}\,dx \approx \frac{0.5}{3}\left(\sqrt{2} + 4\sqrt{4.375} + 2(3) + 4\sqrt{16.625} + 2\sqrt{28} + 4\sqrt{43.875} + \sqrt{65}\right)$$

$$\approx \boxed{12.871}$$

Both methods discussed above are only approximation to the desired area.we do not get the actual area but an area which is very close to actual one.if we calculate a desired area by different methods then it is Simpsons one third rule which gives best results.