Spline Interpolation

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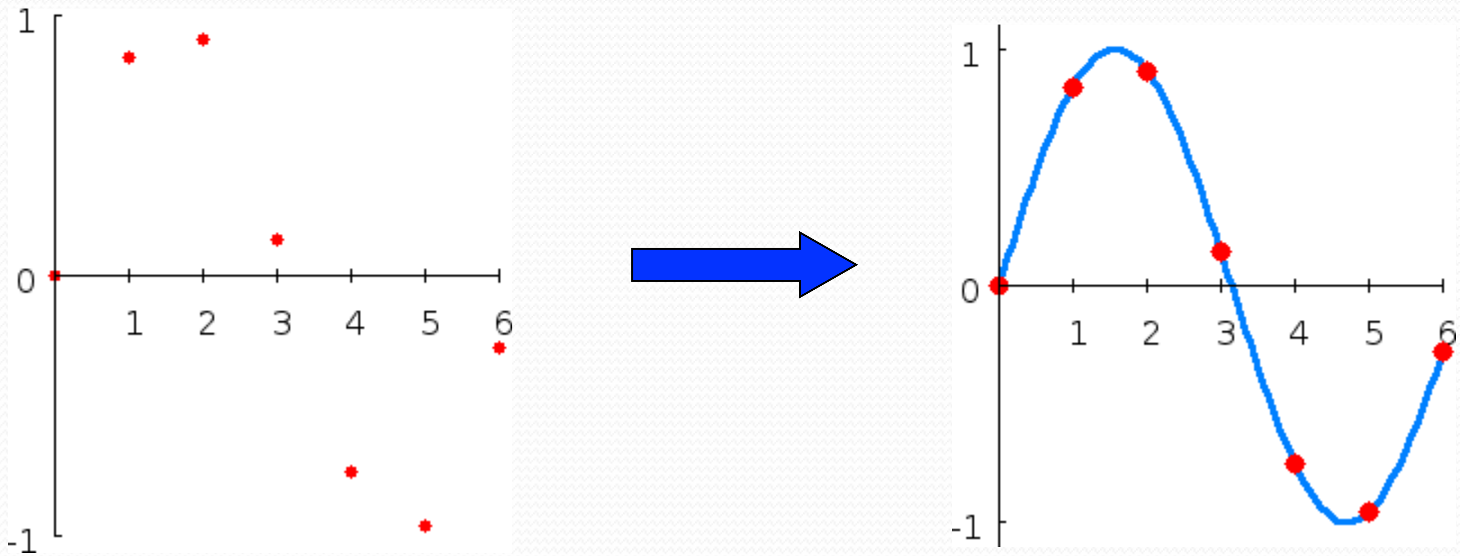
Interpolation

A method of constructing a function that crosses through a discrete set of known data points.



Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate.

Spline Interpolation

- There are cases where polynomial interpolation is bad
- Noisy data
- Sharp Corners (slope discontinuity)
- Humped or Flat data

- Overshoot
- Oscillations

Why Splines ?

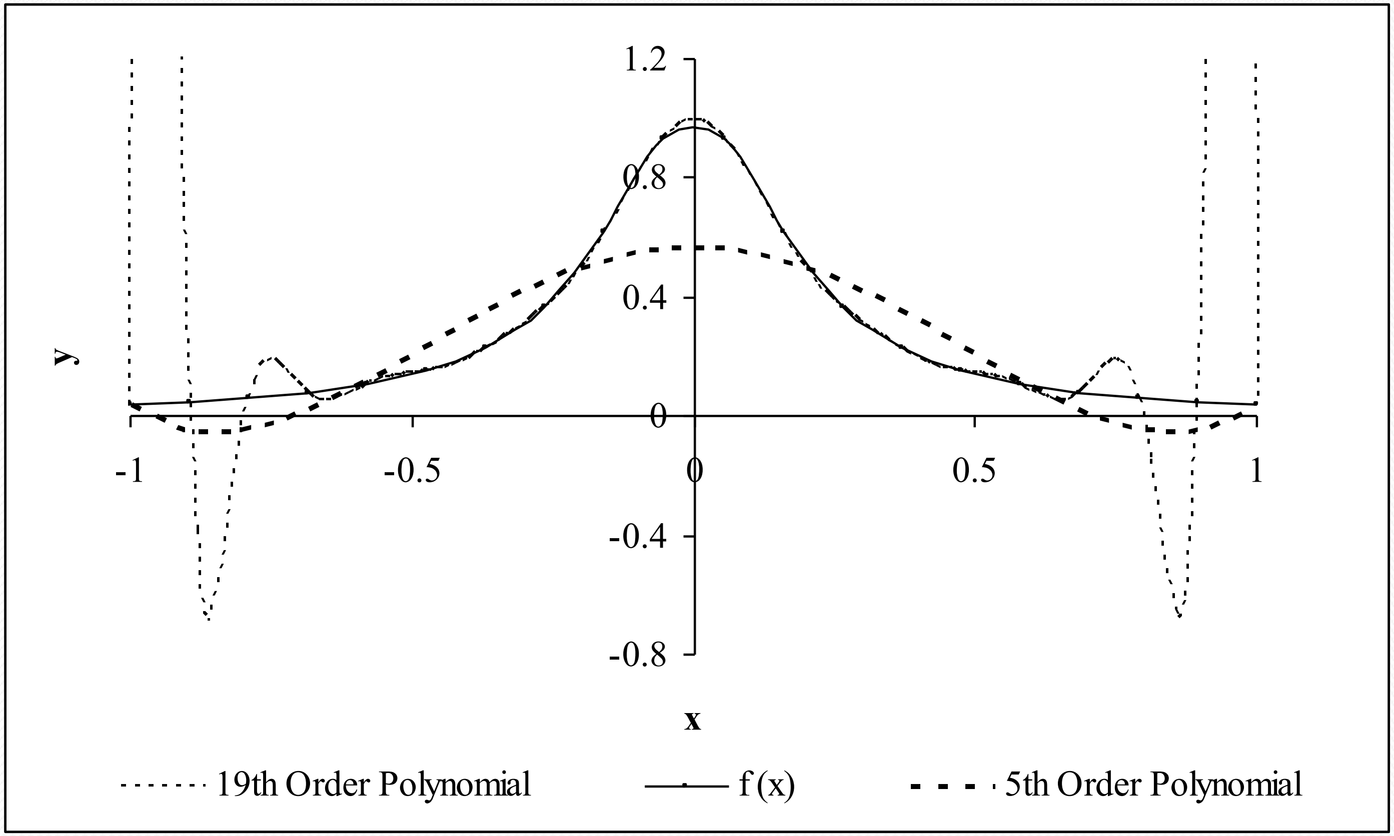


Figure : Higher order polynomial interpolation is a bad idea

Spline Interpolation

Idea behind splines

- **Use lower order polynomials to connect subsets of data points**
- **Make connections between adjacent splines smooth**
- **Lower order polynomials avoid oscillations and overshoots**

Spline Interpolation Definition

- Given n+1 distinct **knots** x_i such that:

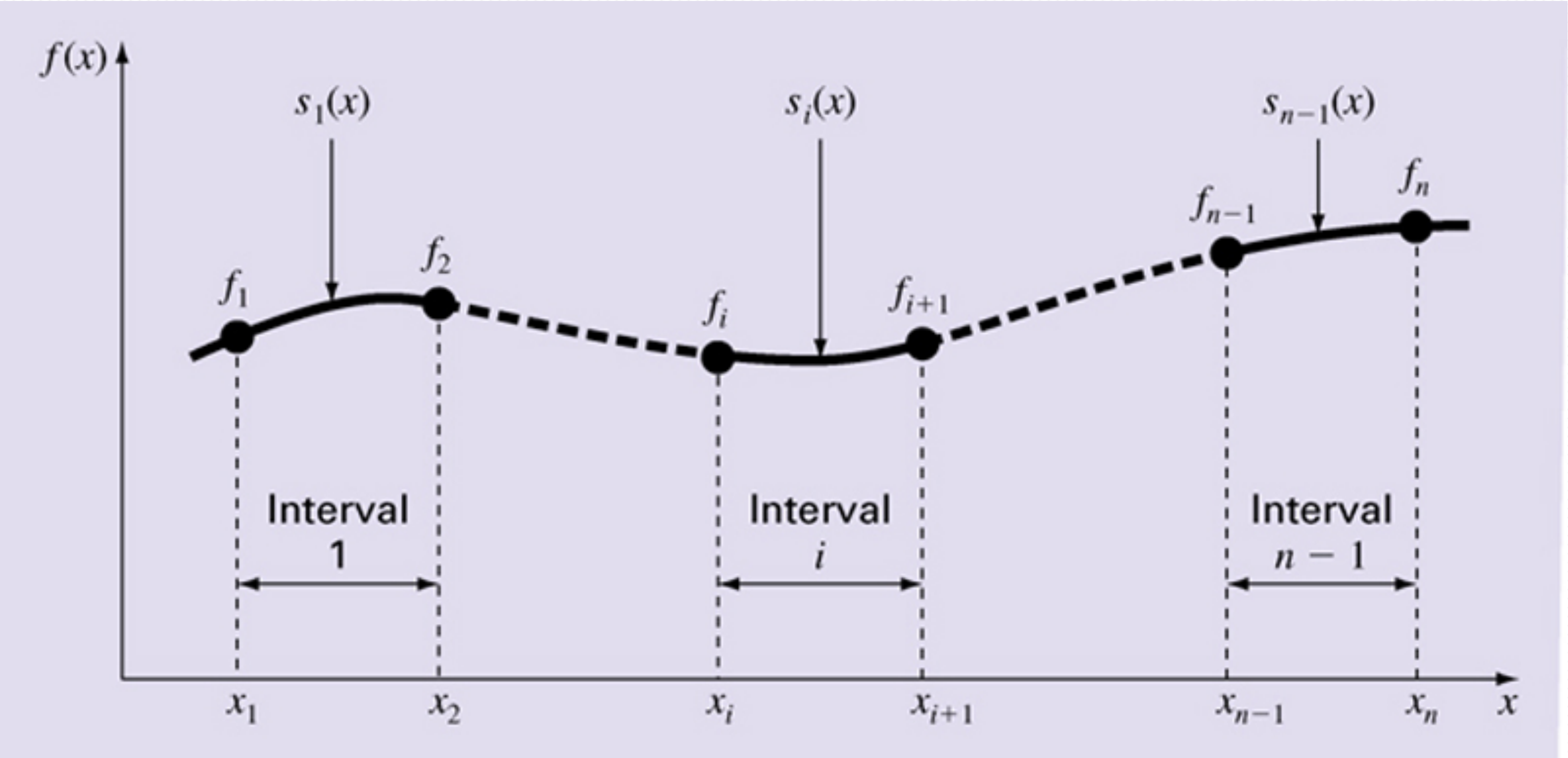
$$x_0 < x_1 < \ldots < x_{n-1} < x_n,$$

with n+1 **knot values** $\boldsymbol{y_i}$ find a spline function

$$S(x) := \begin{cases} S_0(x) & x \in [x_0, x_1] \\ S_1(x) & x \in [x_1, x_2] \\ \vdots & \vdots \\ S_{n-1}(x) & x \in [x_{n-1}, x_n] \end{cases}$$

with each $S_i(x)$ a polynomial of degree at most n.

Splines



- There are n-1 intervals and n data points
- $s_i(x)$ is a piecewise low-order polynomial

(a) Linear spline

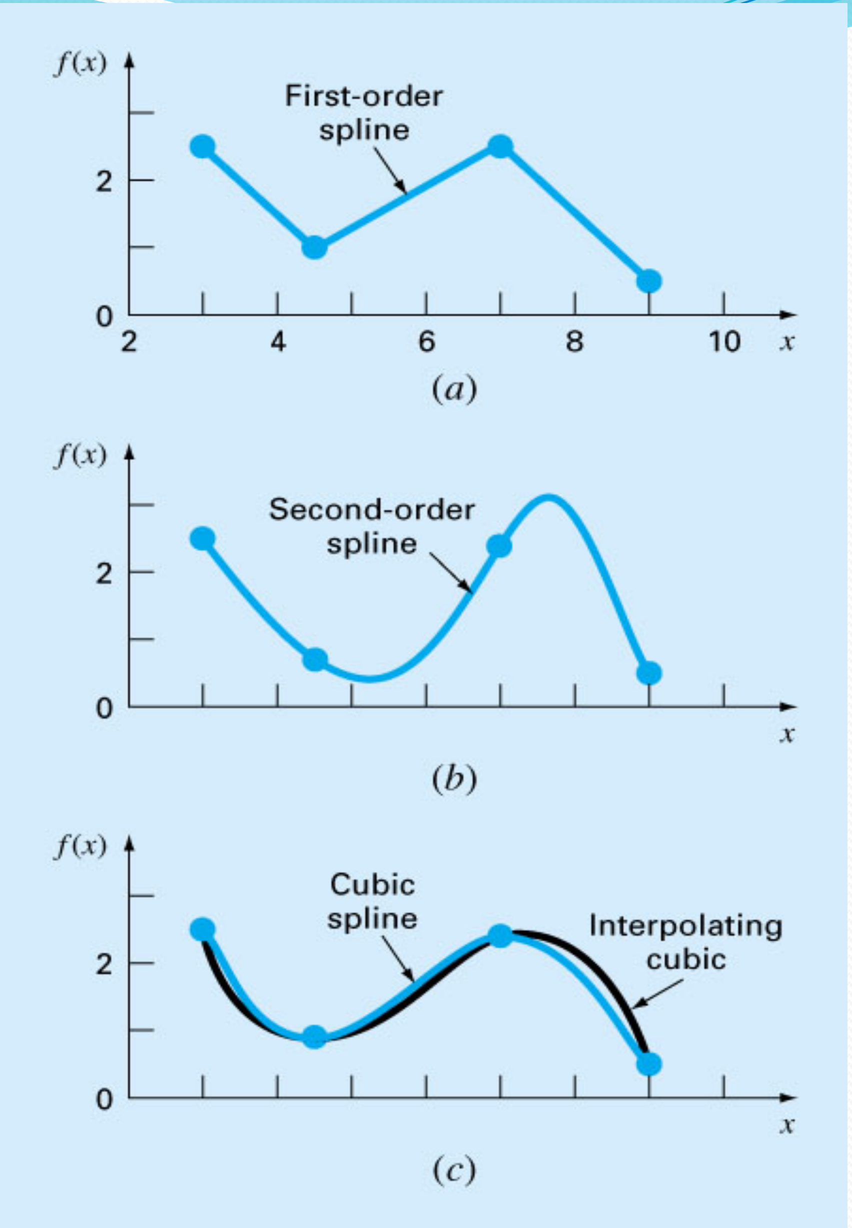
- Derivatives are not continuous
- Not smooth

(b) Quadratic spline

- Continuous 1st derivatives

(c) Cubic spline

- Continuous 1st & 2nd derivatives
- Smoother



Linear splines

- Connect each two points with straight line
- Functions connecting each pair of points are

$$s_1(x) = a_1 + b_1(x - x_1) \; ; \qquad x_1 \le x \le x_2$$

$$s_2(x) = a_2 + b_2(x - x_2) \; ; \qquad x_2 \le x \le x_3$$

M

$$s_i(x) = a_i + b_i(x - x_i) \; ; \qquad x_i \le x \le x_{i+1}$$

M

$$s_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) \; ; \qquad x_{n-1} \le x \le x_n$$

- **slope**

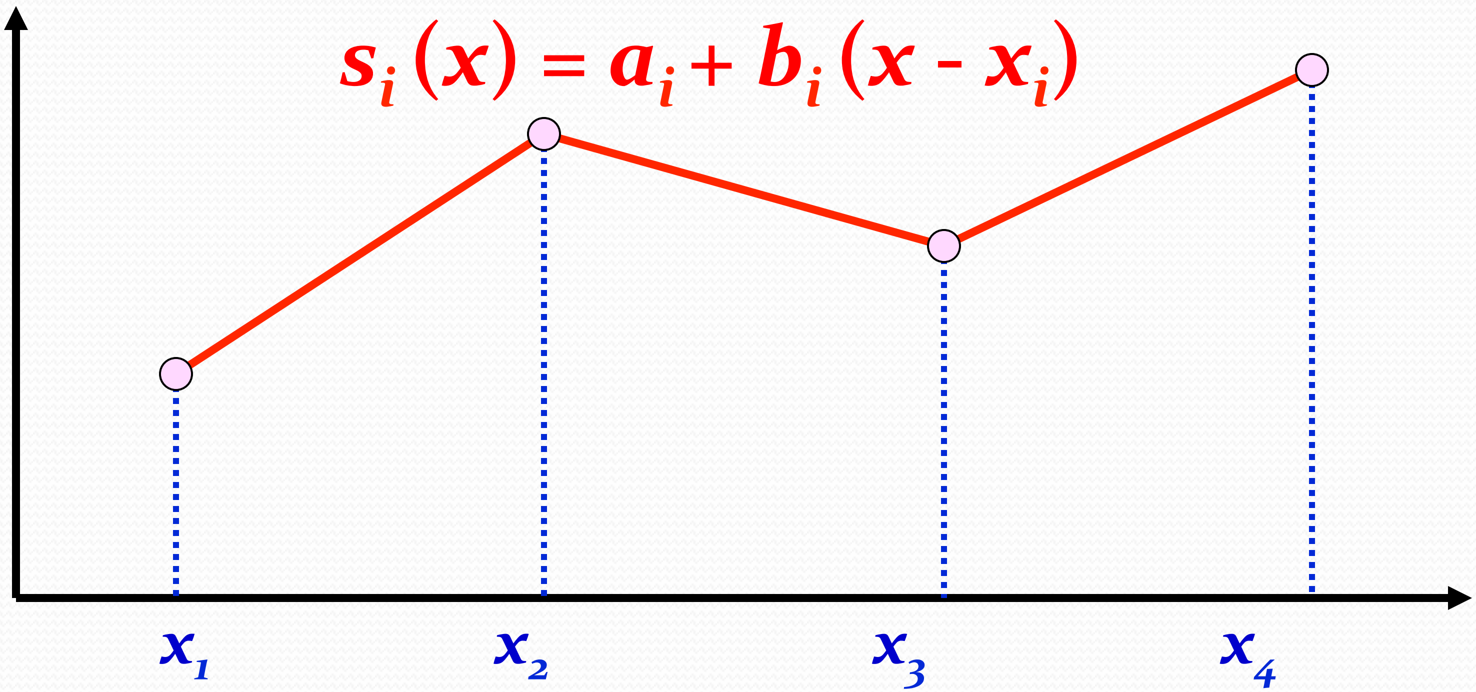
$$b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

Linear Splines

$$data\ points: (x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$$

$$interval: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$

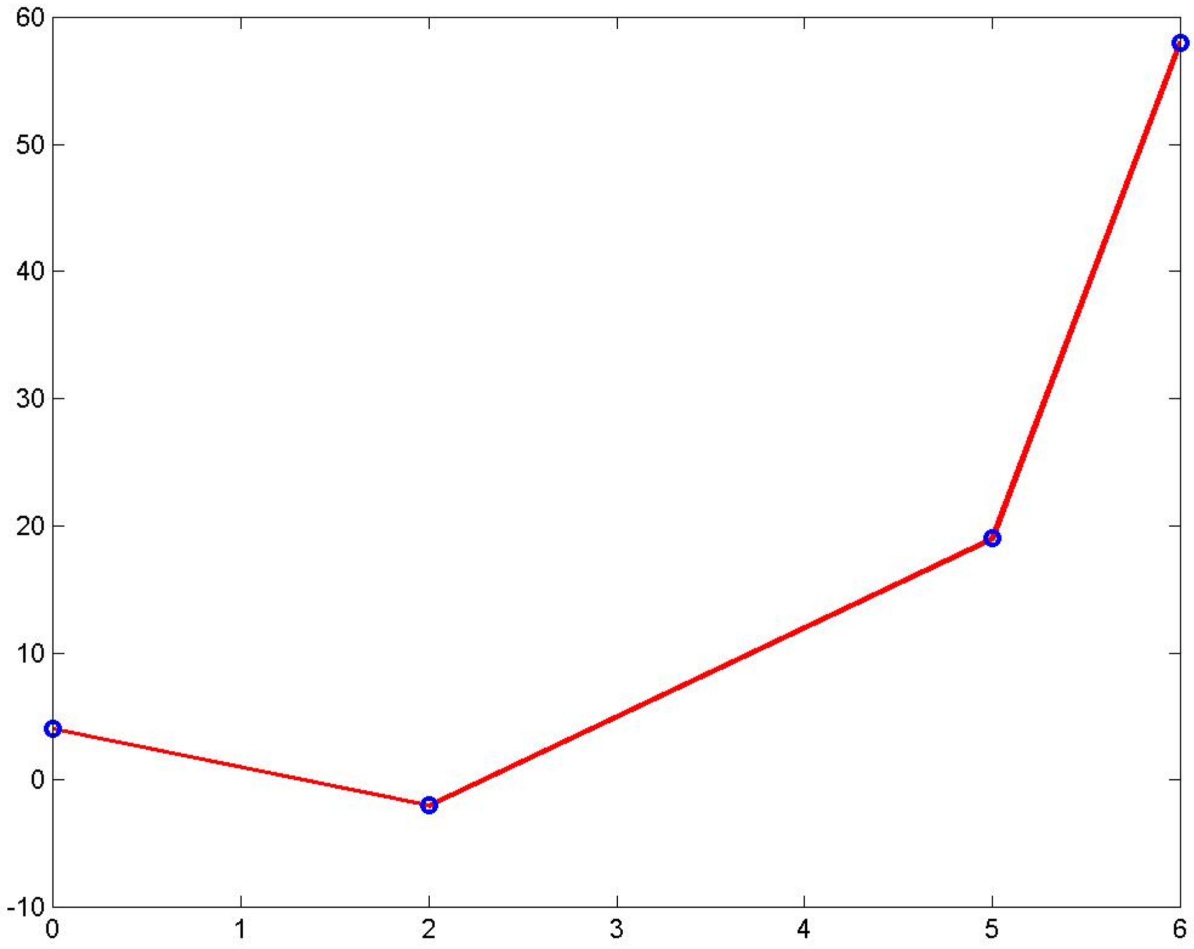
$$s_i(x) = a_i + b_i(x - x_i)$$



Linear splines are exactly the same as linear interpolation!

Example:

x	$f(x)$	b_i
0	4	
		−3
2	−2	
		7
5	19	
		39
6	58	



Linear Splines

- **Problem with linear splines -- not smooth at data points (or knots)**
- **First derivative (slope) is not continuous**
- **Use higher-order splines to get continuous derivatives**
- **Equate derivatives at neighboring splines**
- **Continuous functional values and derivatives**

Quadratic Spline

- Spline of Degree 2

- A function Q is called a spline of degree 2 if
  - The domain of Q is an interval $[a, b]$.
  - Q and Q' are continuous functions on $[a, b]$.
  - There are points x_i (called knots) such that $a = x_0 < x_1 < \ldots < x_n = b$ and Q is a polynomial of degree at most 2 on each subinterval $[x_i, x_{i+1}]$.

- A quadratic spline is a continuously differentiable piecewise quadratic function.

Quadratic Splines

Quadratic splines - continuous first derivatives

May have discontinuous second and higher derivatives

Derive second order polynomial between each pair of points $f_i(x) = a_i x^2 + b_i x + c_i$

For n points (*i=1,…,n*): (n-1) intervals & 3 (n-1) unknown parameters (*a's*, *b's*, and *c's*)

Need 3(n-1) equations

Exercise

- Which of the following is a quadratic spline?

$$A(x) = \begin{cases} x^2 & x \in [-2,0] \\ x^2 + 2x & x \in [0,1] \\ x^2 + x + 1 & x \in [1,2] \end{cases}$$

$$B(x) = \begin{cases} x^2 & x \in [-2,0] \\ -x^2 & x \in [0,1] \\ 1 - 2x & x \in [1,2] \end{cases}$$

Exercise (Solution)

$$A(x) = \begin{cases} x^2 & x \in [-2,0] \\ x^2 + 2x & x \in [0,1] \\ x^2 + x + 1 & x \in [1,2] \end{cases}$$

$A_0'(x) = 2x$

$A_1'(x) = 2x + 2$

At $x = 0$, $A_0'(x) \neq A_1'(x)$.

Thus $A(x)$ is not a quadratic spline

$$B(x) = \begin{cases} x^2 & x \in [-2,0] \\ -x^2 & x \in [0,1] \\ 1 - 2x & x \in [1,2] \end{cases}$$

$B_0(0) = 0 = B_1(0)$

$B_1(1) = -1 = B_2(1)$

$B_0'(0) = 0 = B_1'(0)$

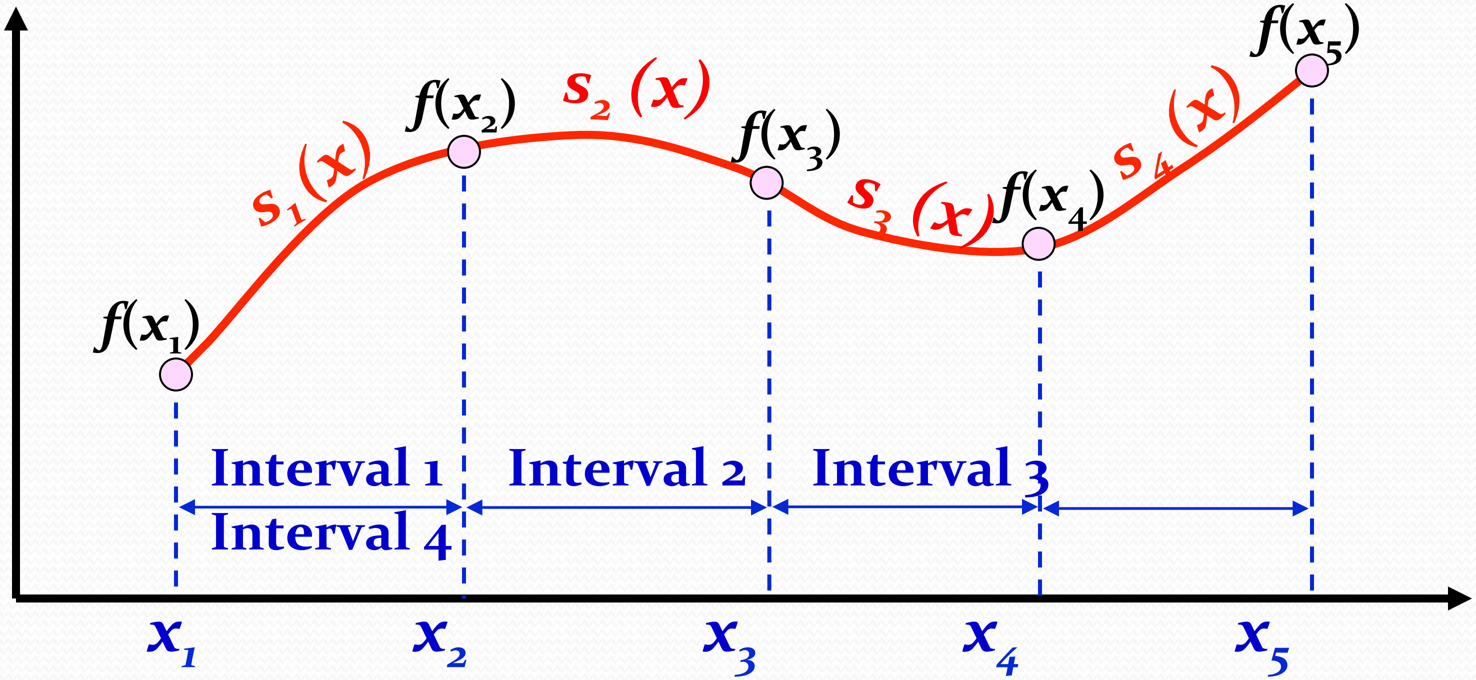
$B_1'(1) = -2 = B_2'(1)$

Since B and B' are continuous for all x in [-2,2], and since B_i's are all polynomial of degree ≤ 2, B is a quadratic spline.

Quadratic Splines

$$\text{data points}: (x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$$

$$\text{interval}: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$



$s_i(x) = a_i + b_i (x-x_i) + c_i(x-x_i)^2$ **3(n-1) unknowns**

Piecewise Quadratic Splines

$$\begin{cases} on\left[x_1,x_2\right]: & s_1(x)=a_1+b_1(x-x_1)+c_1(x-x_1)^2 \\ on\left[x_2,x_3\right]: & s_2(x)=a_2+b_2(x-x_2)+c_2(x-x_2)^2 \\ on\left[x_3,x_4\right]: & s_3(x)=a_3+b_3(x-x_3)+c_3(x-x_3)^2 \\ on\left[x_4,x_5\right]: & s_4(x)=a_4+b_4(x-x_4)+c_4(x-x_4)^2 \end{cases}$$

Example with $n = 5$

1. **The function must pass through all the points x_i (continuity condition)**
2. **The function values of adjacent polynomials must be equal at all nodes (identical to condition 1.)** $2(n-2) + 2 = 2(n-1)$
3. **The 1st derivatives are continuous at interior nodes x_i, $(n-2)$**
4. **Assume that the second derivatives is zero at the first points**

$$2(n-1) + (n-2) + (1) = 3(n-1)$$

$3(n-1)$ equations for $3(n-1)$ unknowns

Observations

- $n+1$ points
- n intervals
- Each interval is connected by a 2nd-order polynomial $Q_i(x) = a_i x^2 + b_i x + c_i$, $i = 0, \ldots, n-1$.
- Each polynomial has 3 unknowns
- Altogether there are $3n$ unknowns
- Need $3n$ equations (or conditions) to solve for $3n$ unknowns

1. Interpolating conditions
   - On each sub interval $[x_i, x_{i+1}]$, the function $Q_i(x)$ must satisfy the conditions

     $Q_i(x_i) = f(x_i)$ and $Q_i(x_{i+1}) = f(x_{i+1})$

   - These conditions yield $2n$ equations

$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

$$a_i x_{i+1}^2 + b_i x_{i+1} + c_i = f(x_{i+1})$$

$$i = 0, \ldots, n-1$$

2. Continuous first derivatives

- The first derivatives at the interior knots must be equal.
- This adds n-1 more equations:

$$2a_i x_i + b_i = 2a_{i+1} x_i + b_{i+1} \qquad i = 1, \ldots, n-1$$

We now have $2n + (n - 1) = 3n - 1$ equations.

We need one more equation.

3. Assume the 2^{nd} derivatives is zero at the first point.

- This gives us the last condition as

$$2a_1 = 0 \Rightarrow a_1 = 0$$

– With this condition selected, the first two points are connected by a straight line.

Example

i	0	1	2	3
x_i	3	4.5	7	9
$f(x_i)$	2.5	1	2.5	0.5

- Fit quadratic splines to the given data points.

Example (Solution)

1. Interpolating conditions

$$
\begin{aligned}
9a_1 &+ 3b_1 &+ c_1 &= 2.5 \\
20.25a_1 &+ 4.5b_1 &+ c_1 &= 1.0 \\
20.25a_2 &+ 4.5b_2 &+ c_2 &= 1.0 \\
49a_2 &+ 7b_2 &+ c_2 &= 2.5 \\
49a_3 &+ 7b_3 &+ c_3 &= 2.5 \\
81a_3 &+ 9b_3 &+ c_3 &= 0.5
\end{aligned}
$$

2. Continuous first derivatives

$$
\begin{aligned}
9a_1 + b_1 &= 9a_2 + b_2 \\
14a_2 + b_2 &= 14a_3 + b_3
\end{aligned}
$$

3. Assume the 2nd derivatives is zero at the first point.

$$a_1 = 0$$

Example (Solution)

We can write the system of equations in matrix form as

$$\begin{bmatrix} 9 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 20.25 & 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\ 9 & 1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 14 & 1 & 0 & -14 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 1 \\ 1 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Notice that the coefficient matrix is sparse.

Example (Solution)

The system of equations can be solved to yield

$$
\begin{array}{lll}
a_1 = 0 & b_1 = -1 & c_1 = 5.5 \\
a_2 = 0.64 & b_2 = -6.76 & c_2 = 18.46 \\
a_3 = -1.6 & b_3 = 24.6 & c_3 = -91.3
\end{array}
$$

Thus the quadratic spline that interpolates the given points is

$$
Q(x) = \begin{cases} -x + 5.5 & x \in [3,4.5] \\ 0.64x^2 - 6.76x + 18.46 & x \in [4.5,7] \\ -1.6x^2 + 24.6x - 91.3 & x \in [7,9] \end{cases}
$$

Natural Cubic Spline Interpolation

SPLINE OF DEGREE k = 3

- The domain of S is an interval [a,b].
- S, S', S'' are all continuous functions on [a,b].
- There are points t_i (the knots of S) such that $a = t_o < t_1 < .. t_n = b$ and such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$.

x	t_0	t_1	…	t_n
y	y_0	y_1	…	y_n

t_i are knots

Natural Cubic Spline Interpolation

$$S(x) = \begin{cases} S_0(x), & x \in [x_0, x_1] \\ S_1(x), & x \in [x_1, x_2] \\ \cdots \\ S_{n-1}(x), & x \in [x_{n-1}, x_n] \end{cases}$$

$S_i(x)$ is a cubic polynomial that will be used on the subinterval [$\boldsymbol{x_i}$, $\boldsymbol{x_{i+1}}$].

Cubic Splines

Cubic splines avoid the straight line and the over-swing

$$f_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

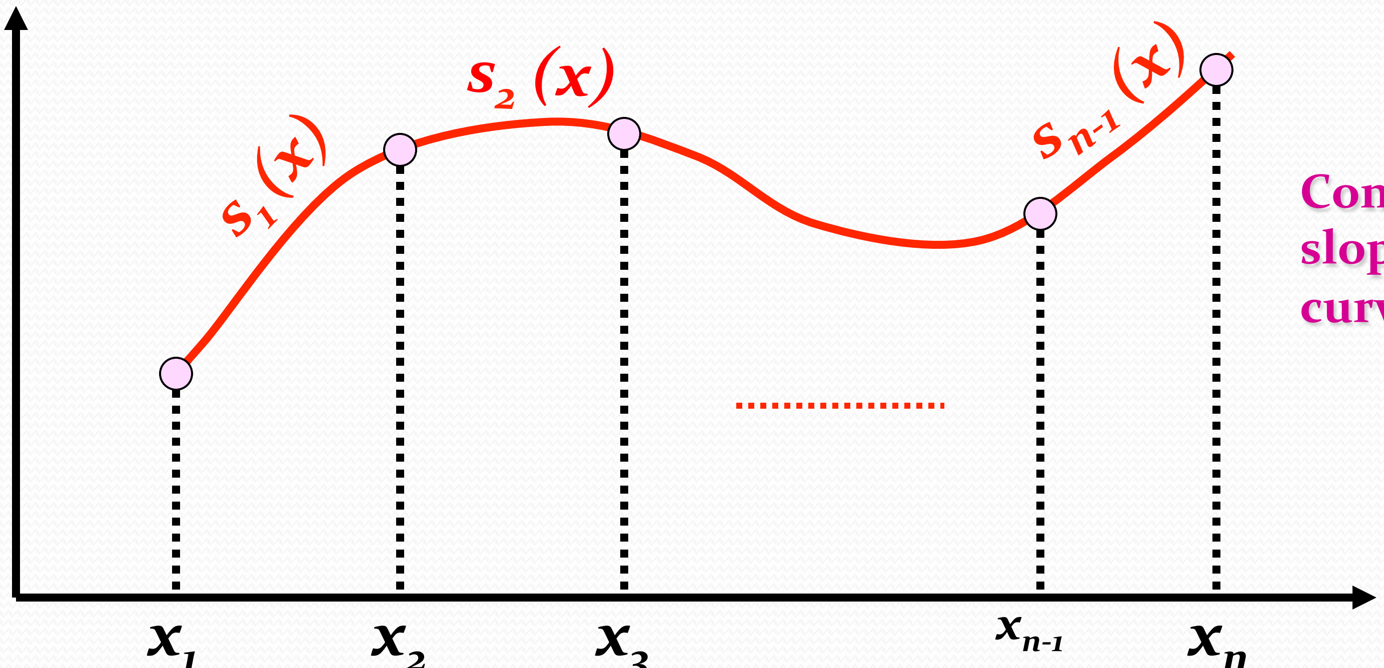
Can develop method like we did for quadratic – 4(n–1) unknowns – 4(n–1) equations

- interior knot equality
- end point fixed
- interior knot first derivative equality
- assume derivative value if needed

Piecewise Cubic Splines

$$data\ points: (x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$$

$$interval: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$



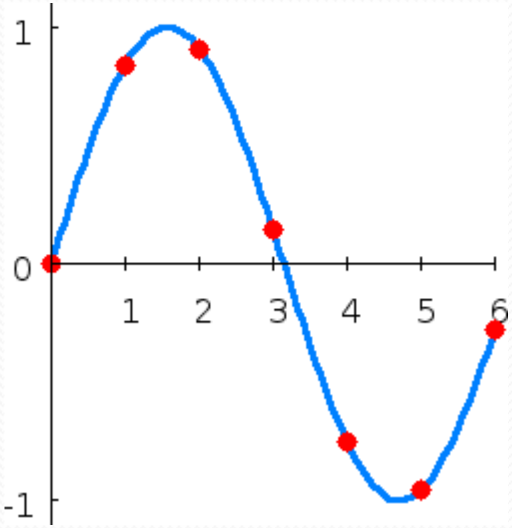
Continuous slopes and curvatures

$$f_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

4(n-1) unknowns

Natural Cubic Spline Interpolation

- $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$
  - 4 Coefficients with n subintervals = 4n equations
  - There are 4_{n-2} conditions
    - Interpolation conditions
    - Continuity conditions
    - Natural Conditions
  - $S''(x_o) = 0$
  - $S''(x_n) = 0$



Cubic Spline (4n conditions)

1. Interpolating conditions (2n conditoins).
2. Continuous 1st derivatives (n-1 conditions)
   - The 1st derivatives at the interior knots must be equal.
3. Continuous 2nd derivatives (n-1 conditions)
   - The 2nd derivatives at the interior knots must be equal.
4. Assume the 2nd derivatives at the end points are zero (2 conditions).
   - This condition makes the spline a "natural spline".

Summary

- Advantages of spline interpolation over polynomial interpolation

- The conditions that are used to derive the quadratic and cubic spline functions

- Characteristics of cubic spline
  - Overcome the problem of "overshoot"
  - Easier to derive (than high-order polynomial)
  - Smooth (continuous 2^{nd}-order derivatives)

THANK YOU!!