

Maximal mixing as a ‘sum’ of small mixings

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In models with two sources of neutrino masses, we look at the possibility of generating maximal/large mixing angles in the total mass matrix, where both the sources have only small mixing angles. We show that in the two generation case, maximal mixing can naturally arise only when the total neutrino mass matrix has a quasi-degenerate pattern. The best way to demonstrate this is by decomposing the quasi-degenerate spectrum into hierarchical and inverse-hierarchical mass matrices, both with small mixing. Such a decomposition of the quasi-degenerate spectra is in fact very general and can be done irrespective of the mixing present in the mass matrices. With three generations, and two sources, we show that only one or all the three small mixing angles in the total neutrino mass matrix can be converted to maximal/large mixing angles. The decomposition of the degenerate pattern in this case is best realised into sub-matrices whose dominant eigenvalues have an alternating pattern. On the other hand, it is possible to generate two large and one small mixing angle if either one or both of the sub-matrices contain maximal mixing. We present example textures of this. With three sources of neutrino masses, the results remain almost the same as long as all the sub-matrices contribute equally. The Left-Right Symmetric model where Type I and Type II seesaw mechanisms are related provides a framework where small mixings can be converted to large mixing angles, for degenerate neutrinos.

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I. INTRODUCTION

While neutrino masses have been thoroughly established experimentally [1], the question of how they attain their masses still needs to be understood. Perhaps, the most elegant mechanism of generating neutrino masses is through the seesaw mechanism [2]. Here one trades the tininess of the neutrino masses with high scale Majorana masses for right-handed neutrinos introduced for this purpose. While the original seesaw mechanism dealt only with right-handed heavy neutrino states, in recent years, it has been realized that there could be other heavy triplet scalars [3] or even triplet fermions [4, 5] which could play the same role as right-handed neutrinos in the original seesaw mechanism. These mechanisms are named as Type I, Type II and Type III seesaw mechanisms respectively (for recent reviews please see [6, 7]). While one of the three seesaw mechanisms suffices to generate non-zero neutrino masses, it is interesting to note that in most Grand Unified Theory (GUT) models, there is more than one seesaw mechanism at work. For example, in SO(10) models both Type I and Type II seesaw mechanisms are simultaneously present as soon as one considers representations of the type $\overline{126}$ [8]. In Left-Right Symmetric (LRS) models, the Type I and Type II seesaw mechanisms are not just present, but they are also related to each other [9]. Similarly, Type I and Type III mechanisms co-exist in SU(5) model with an adjoint fermion representation [5]. In most of these investigations, typically one considers one of them to be dominant while the other to be subdominant.

One of the crucial features of seesaw mechanism was its ability to generate large or maximal mixing even though the mixing present in the Dirac neutrino Yukawa couplings is small like in the hadronic sector. In fact this is what typically happens in a SO(10) GUT [10], where neutrino Dirac Yukawa couplings have the same structure as the top Yukawa couplings; even in such cases large mixing in the neutrino sector is possible. However this would require large hierarchies in the masses of the right-handed neutrinos which is in conflict with thermal leptogenesis in these models [11]¹. In the present work, we look for an alternative method to generate large/maximal mixings instead of using the ‘seesaw-effect’. We will use the fact that most models GUT models like SO(10) have more than one seesaw mechanism at work. However, instead of restricting ourselves to any particular GUT model or the seesaw mechanism, we analyze the general situation where there are two sources for neutrino masses and both of these contain small neutrino mixing.

Our analysis shows that the total neutrino mass matrix which is given by the sum of the two neutrino sources can have large or maximal mixing only if the resulting pattern of the neutrino

¹ This is true when the mixing angles in neutrino Dirac Yukawa are exactly like CKM angles.

masses is of the quasi-degenerate form. A crucial condition which needs to be satisfied to reach this conclusion is that the large eigenvalues of the sub-matrices do not cancel in the total mass matrix. This results in the sub-matrices taking the form of hierarchial and inverse-hierarchial matrices whose sum leads to the quasi-degenerate form. Given that the decomposition of the degenerate spectrum in to hierarchial and inverse-hierarchial mass matrices is quite generic, as we will demonstrate here, one can enumerate the possible forms the individual sub-matrices can take. It should be noted that the decomposition itself is independent of the actual mechanism responsible for generating neutrino masses *i.e.*, doesn't depend on whether there is a seesaw mechanism at work or not. It is well known that the quasi-degenerate pattern for neutrino masses can be achieved both with [12] and without [13] seesaw mechanism. However, as will demonstrate later, the model dependence enters, if one wants to realise the decomposition in terms of independent Lagrangian parameters which for example is possible in Type I seesaw mechanism.

The simple example where our scheme of things can be realised is the LRS model where both Type I and Type II seesaw mechanisms are simultaneously present. We will explicitly present the conditions on the LRS parameters required in order to realize the mechanism. The paper is organised as follows. In Sec. II, we analyse the two generation case and show how only when the quasi-degeneracy is satisfied in the final matrix, one can have large or maximal mixing. We also describe all the possible decompositions of the degenerate spectra. We further discuss how this scheme can be incorporated within the LRS models. In Sec. III, we consider two cases (*a*) with two seesaw mechanisms or two sources, and (*b*) with three sources. We then demonstrate the decomposition of the quasi-degenerate spectrum and discuss the subtleties which arise in this case. We also determine the required parameter values within the LRS model for both the cases. We close with summary and outlook in Sec. IV. Generalisation of our result to the case of n sources of neutrino masses is given in Appendix A.

II. LARGE MIXING AS SUM OF SMALL MIXING ANGLES

Consider a model for neutrino masses in which the total neutrino mass matrix is given by

$$\mathbb{M}_\nu = \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)}, \quad (1)$$

where $\mathbb{M}_\nu^{(1)}$ and $\mathbb{M}_\nu^{(2)}$ can be thought of as two individual sources of neutrino mass. For example, $\mathbb{M}_\nu^{(1)}$ could have its origin in Type I seesaw whereas $\mathbb{M}_\nu^{(2)}$ could have its origin in Type II seesaw

mechanism in a model like SO(10) where both these mechanisms are simultaneously present ². Irrespective of their origin, let us assume that both $\mathbb{M}_\nu^{(1)}$ and $\mathbb{M}_\nu^{(2)}$ contain only small mixing angles. We now ask the question whether it is possible to have in the total mass matrix \mathbb{M}_ν (a) maximal or large mixing, and (b) a *reasonable* Δm^2 without fine-tuning. By this we mean, that the Δm^2 is determined in terms of the dominant eigenvalues of $\mathbb{M}_\nu^{(i)}$ (where $i = 1, 2$). To make the discussion concrete, we will stick to two generation case in the present section. Denoting

$$\mathbb{M}_\nu^{(1)} = \begin{pmatrix} m_{ee}^{(1)} & m_{e\mu}^{(1)} \\ m_{e\mu}^{(1)} & m_{\mu\mu}^{(1)} \end{pmatrix}, \quad \mathbb{M}_\nu^{(2)} = \begin{pmatrix} m_{ee}^{(2)} & m_{e\mu}^{(2)} \\ m_{e\mu}^{(2)} & m_{\mu\mu}^{(2)} \end{pmatrix}, \quad (2)$$

we can easily derive the following relations :

$$\tan 2\theta = \frac{2m_{e\mu}^{(1)} + 2m_{e\mu}^{(2)}}{m_{\mu\mu}^{(2)} + m_{\mu\mu}^{(1)} - m_{ee}^{(2)} - m_{ee}^{(1)}} \quad (3)$$

$$= \tan 2\theta^{(1)} \frac{1}{(1+d)} + \tan 2\theta^{(2)} \frac{d}{(1+d)}, \quad (4)$$

where $d = (m_{\mu\mu}^{(2)} - m_{ee}^{(2)}) / (m_{\mu\mu}^{(1)} - m_{ee}^{(1)})$ and $\theta^{(1)}$ and $\theta^{(2)}$ are the mixing angles of $\mathbb{M}_\nu^{(1)}$ and $\mathbb{M}_\nu^{(2)}$ respectively. From this expression, it is obvious that when both the mixing angles, $\theta^{(1)}$ and $\theta^{(2)}$ are small, the only region where θ would be maximal is when $d = -1$. Notice that the small mixing in $\mathbb{M}_\nu^{(i)}$ would mean (a) $2 m_{e\mu}^{(i)} \ll |m_{\mu\mu}^{(i)} - m_{ee}^{(i)}|$, and (b) $m_{\mu\mu}^{(i)} \neq m_{ee}^{(i)}$ for $i = (1, 2)$ i.e, the splitting in the diagonal entries is much larger than the off-diagonal entry such that the mixing remains small. Assuming at least one of the diagonal entries in each of the matrix $\mathbb{M}_\nu^{(i)}$ is large, we have the following three solutions for $d = -1$

$$(A) m_{\mu\mu}^{(2)} = -m_{\mu\mu}^{(1)},$$

$$(B) m_{ee}^{(2)} = -m_{ee}^{(1)}, \text{ and}$$

$$(C) m_{ee}^{(2)} = m_{\mu\mu}^{(1)} \text{ or } m_{\mu\mu}^{(2)} = m_{ee}^{(1)}.$$

The solution of the type (A) would represent the case in which both the matrices $\mathbb{M}_\nu^{(i)}$ are of the hierarchial form with one dominant diagonal element (the $\mu\mu$ entry). However in the total mass matrix \mathbb{M}_ν this entry gets cancelled. To illustrate this, consider the following textures for $\mathbb{M}_\nu^{(i)}$

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} z & x \\ x & 1 + z' \end{pmatrix} + m_2 \begin{pmatrix} 0 & y \\ y & -1 \end{pmatrix}, \quad (5)$$

² In fact, in most models of neutrino masses, one of them, say $\mathbb{M}_\nu^{(1)}$ could correspond to zeroth order mass while the other $\mathbb{M}_\nu^{(2)}$ could correspond to perturbations required to make contact with the experimental results.

where x, y, z are the small entries compared to $m_{\mu\mu}^{(1)}/m_1 \equiv (1+z')$ and $m_{\mu\mu}^{(2)}/m_2 \equiv -1$. Notice that the dominant eigenvalues of $\mathbb{M}_\nu^{(i)}$ have opposite CP parities as the maximal mixing requirement condition is now given as $m_1 \approx m_2 \approx m$. In this limit, the total mass matrix has the form

$$\mathbb{M}_\nu = m \begin{pmatrix} z & x+y \\ x+y & z' \end{pmatrix}. \quad (6)$$

The total mass matrix here has no trace of the dominant element of the $\mathcal{O}(m)$ which was present in the sub-matrices. It has been cancelled in such a way that the condition $m_{\mu\mu}^{(2)} + m_{\mu\mu}^{(1)} = m_{ee}^{(1)} + m_{ee}^{(2)}$ is satisfied, which is the same as the full condition of $d = -1$ which would mean $z' = z$, rather than the sub-condition, (A) $m_{\mu\mu}^{(2)} = -m_{\mu\mu}^{(1)}$, which would instead mean $z' = 0$. The role of the large element of the $\mathcal{O}(m)$ has only been to generate the small mixing in the respective $\mathbb{M}_\nu^{(i)}$. Thus, at the level of total mass matrix \mathbb{M}_ν , the properties are determined by the small entries of the original sub-matrices. The mass-squared splitting Δm^2 of the total mass matrix in terms of the elements of $\mathbb{M}_\nu^{(i)}$ is given by

$$\Delta m^2 = (m_{ee}^{(1)} + m_{\mu\mu}^{(1)} + m_{ee}^{(2)} + m_{\mu\mu}^{(2)}) \sqrt{(m_{\mu\mu}^{(1)} + m_{\mu\mu}^{(2)} - m_{ee}^{(1)} - m_{ee}^{(2)})^2 + 4(m_{e\mu}^{(1)} + m_{e\mu}^{(2)})^2}, \quad (7)$$

which reduces in the present case to

$$\Delta m^2 = m^2(z+z') \sqrt{4(x+y)^2 + (z-z')^2}. \quad (8)$$

In the limit $z' \rightarrow 0$, the mixing $\tan 2\theta = 2(x+y)/z$ would depend on the relative magnitudes of x, y and z with large mixing being possible as long as $x+y \gg z$. Similarly, in the limit $z' \approx z$, maximal/large mixing is possible, and a hierarchical pattern for the neutrinos can arise if $x, y \sim z, z'$. Solutions of the class (B) also lead to similar results with the dominant entries of the sub-matrices $\mathbb{M}_\nu^{(i)}$ being cancelled in the total mass matrix. We do not find these solutions attractive as large mixing can only come when the dominant elements (' ee ' elements in this case) cancel precisely to such an extent to be equal to the sum of the other diagonal elements (' $\mu\mu$ ' elements). We now go on to discuss the solutions (C) which we find more natural.

The solutions of the type (C) are given by $m_{ee}^{(2)} = m_{\mu\mu}^{(1)}$ or $m_{\mu\mu}^{(2)} = m_{ee}^{(1)}$. The condition now requires that the opposite diagonal elements of the sub-matrices are equal. This naturally sets the $\mathbb{M}_\nu^{(i)}$ to have an opposite ordering of their eigenvalues *i.e.*, one with normal hierarchy and the other has inverse hierarchy. For illustration, let us consider the (sub)-case with $m_{\mu\mu}^{(2)} = m_{ee}^{(1)}$. This can

be represented as

$$\begin{aligned} \mathbb{M}_\nu &= \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)} = m_1 \begin{pmatrix} 0 & x \\ x & 1 \end{pmatrix} + m_2 \begin{pmatrix} 1 & -y \\ -y & 0 \end{pmatrix} \\ &= \begin{pmatrix} m_2 & m_1x - m_2y \\ m_1x - m_2y & m_1 \end{pmatrix}, \end{aligned} \quad (9)$$

where x, y are small entries with $m_{\mu\mu}^{(1)} \equiv m_1$ and $m_{ee}^{(2)} \equiv m_2$. Note that here too as in the earlier case the mixing angles in the individual sub-matrices are small, $\theta \simeq x$ or y , where as the total mixing matrix is given by

$$\tan 2\theta = \frac{2(m_1x - m_2y)}{m_1 - m_2}. \quad (10)$$

In the limit of exact degeneracy between m_1 and m_2 , the mixing is maximal as is evident. However, an important assumption is that both the m_1 and m_2 carry the same sign or equivalently have the same CP parity³. In a more general situation, say when the zeros of the matrices on the RHS are of Eq. (9) are filled with small entries ('ee' element in $\mathbb{M}_\nu^{(1)}$ and ' $\mu\mu$ ' element in $\mathbb{M}_\nu^{(2)}$), the condition for the large mixing is given by $|m_1 - m_2| < 2(m_1x - m_2y)$. Thus, the splitting in the diagonal entries should be much smaller than the off-diagonal elements. The spectrum of the total mass matrix points towards a quasi-degenerate pattern. The eigenvalues are given by :

$$\lambda_{1,2} = \frac{1}{2} \left[m_1 + m_2 \mp \sqrt{(m_1 - m_2)^2 + 4(m_1x - m_2y)^2} \right], \quad (11)$$

which in the limit $m_1 \approx m_2 \approx m$ take the form $m - \epsilon, m + \epsilon$, with $\epsilon = m(x - y)$ being the order of the off-diagonal entry. The $\Delta m^2 = 4m\epsilon$. The other solution of (C), $m_{ee}^{(2)} = m_{\mu\mu}^{(1)}$, corresponds to an interchange of m_1 and m_2 and would lead to similar conclusions.

Finally, let us consider a class of solutions with two large diagonal entries in each of the $\mathbb{M}_\nu^{(i)}$. However given that the mixing in each of them is small, as per the discussion above, the splitting between the diagonal elements should be larger than the off-diagonal entry. This can be parameterised by the following set of matrices :

$$\begin{aligned} \mathbb{M}_\nu &= \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)} = m_1 \begin{pmatrix} 1 + \rho & x \\ x & 1 \end{pmatrix} + m_2 \begin{pmatrix} 1 & x' \\ x' & 1 + \rho' \end{pmatrix} \\ &= \begin{pmatrix} m_1(1 + \rho) + m_2 & m_1x + m_2x' \\ m_1x' + m_2y & m_1 + m_2(1 + \rho') \end{pmatrix}, \end{aligned} \quad (12)$$

³ If the CP parities are opposite the mixing will remain small.

where x, x' are small entries compared to one as before and ρ, ρ' are chosen such that $2|x/\rho| \ll 1$ and $2|x'/\rho'| \ll 1$ to keep the mixing small in \mathbb{M}_1^ν . This would mean a relative hierarchy of the elements in the individual matrices, $m_{ee}^{(1)} \gg m_{\mu\mu}^{(1)} \gg m_{e\mu}^{(1)}$ in \mathbb{M}_1^ν and $m_{\mu\mu}^{(2)} \gg m_{ee}^{(2)} \gg m_{e\mu}^{(2)}$ in \mathbb{M}_2^ν , which is very similar to the case of solutions (C) with a large m_{ee} ($m_{\mu\mu}$) in $\mathbb{M}_\nu^{(1)}$ ($\mathbb{M}_\nu^{(2)}$). Qualitatively, these could form a different class of solutions compared to type (C) with each of the sub-matrices here forming a quasi-degenerate pair with small mixing. However, notice that the total mixing is now given by $\tan 2\theta \approx (x + x')/(\rho' - \rho)$ would remain small as $x, x' \ll \rho, \rho'$ unless $\rho = \rho'$. With this *additional* condition, this class of solutions again falls in to the class (C) *i.e.*, $m_{ee}^{(2)} = m_{\mu\mu}^{(1)}$ or $m_{\mu\mu}^{(2)} = m_{ee}^{(1)}$. However, to distinguish from the solutions in Eq. (9), we will call the class of solutions represented by Eq. (12) as type (C1) ⁴.

In summary, the sum of two mass matrices with small mixing angles would naturally lead to a degenerate spectrum with maximal/large mixing provided we insist there are no cancellations of the large eigenvalues of the individual sub-matrices. The individual sub-matrices could be (a) ordered as hierarchial + inverse-hierarchial with small mixing (solutions of type (C)) or (b) be quasi-degenerate themselves but with small mixing (C1). However as we have seen, solutions of the type (C1) require further precise cancellation in the differences of their large diagonal elements. For this reason, we consider solutions of the type (C) *i.e.*, matrices as parameterised in Eq. (9) to be the most natural. Thus to convert *one* small mixing angle in two matrices to *one* maximal mixing in the total matrix, we would require a *pair* of (quasi)-degenerate eigenvalues with the same CP parities, ordered oppositely in the sub-matrices. This count would be useful when we extend this *degeneracy induced large mixing* to three generations.

A. Decomposition of the Degenerate Spectrum

In the previous section we have seen that a quasi-degenerate pattern naturally emerges if two mass matrices of small mixing are added and we demand large mixing in the total mass matrix. One can instead reverse the argument and might say that the quasi-degenerate spectrum with large mixing can be decomposed in to two matrices with small mixing. In fact, the decomposition of the quasi-degenerate spectrum in to two matrices is more generic and is independent of the mixing present in them. This can be easily be demonstrated by considering zeroth order neutrino mass matrices in the flavour basis. Let us denote the neutrino mass matrix in the flavour basis by

$$\mathbb{M}_\nu = \mathbb{U}_{PMNS} \mathbb{M}_{diag} \mathbb{U}_{PMNS}^\dagger, \quad (13)$$

⁴ Solutions with three large entries in each sub-matrix violate the small mixing assumption.

where $\mathbb{U}_{PMNS} = \mathbb{U}_l^\dagger \mathbb{U}_\nu$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) unitary leptonic mixing matrix. $\mathbb{U}_{PMNS} = \mathbb{U}_\nu$ in a basis in which charged lepton mass matrix is diagonal, *i.e.*, $\mathbb{U}_l = \mathbb{I}_{n \times n}$. In Table I, we have listed the zeroth order mass matrices for hierarchal, inverse-hierarchal and degenerate spectra for the case of small mixing and maximal mixing. In writing down these textures, we have followed Altarelli and Feruglio [14] method, where each of these (zeroth order) mass matrices has to be multiplied by a mass scale m representing the heaviest eigenvalue of the mass matrix. From Table I we can see that, as we go along each column, the degenerate mass matrices \mathbb{C}_i can be expressed as a sum of hierarchal, \mathbb{A} and inverse hierarchal, \mathbb{B} matrices. For example, $\mathbb{C}_0 = \mathbb{A} + \mathbb{B}$, $\mathbb{C}_1 = \mathbb{A} - \mathbb{B}$, $\mathbb{C}_2 = \mathbb{B} - \mathbb{A}$. Note that the mass scale m multiplying \mathbb{C}_i now multiplies both \mathbb{A} and \mathbb{B} . These equations hold irrespective of the mixing being small or maximal. Thus every degenerate mass matrix can be expressed a sum (or difference) of a hierarchial and inverse-hierarchial mass matrices, but with common mass scale given by the degenerate mass m , which is an obvious observation if one just sees the diagonal eigenvalues of each mass matrix in the first column.

Mixing \Rightarrow \mathbb{M}_{diag}	Small \mathbb{X}_ϵ	Maximal \mathbb{X}_M
Hierarchial \mathbb{A} : Diag[0,1]	$\begin{pmatrix} 0 & \epsilon \\ \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$
Inverse hierarchial \mathbb{B} : Diag[1,0]	$\begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
Degenerate \mathbb{C}_0 : Diag[1,1] \mathbb{C}_1 : Diag[-1,1] \mathbb{C}_2 : Diag[1,-1]	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & 2\epsilon \\ 2\epsilon & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & -2\epsilon \\ -2\epsilon & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

TABLE I: Zeroth order textures for small and maximal mixing (setting m_1 and m_2 as dimensionless quantities which are either zero or one depending on the different cases listed) for the two-generation case.

Let us now turn to the question of mixing for the degenerate cases mentioned above. The mixing in $\mathbb{C}_0 = \mathbb{A} + \mathbb{B}$ is undetermined as it is proportional to the identity matrix. This is also the exact degeneracy limit. This situation arises if the mixing angles of \mathbb{A} and \mathbb{B} are not only small, but are also equal. On the other hand, the mixing in $\mathbb{C}_1 = \mathbb{A} - \mathbb{B}$ can be maximal again as we

explained above in the previous section. The mixing in \mathbb{C}_1 and \mathbb{C}_2 will remain small as they have opposite CP parities. An important exception to generate large mixing in terms of small mixing angles through quasi-degeneracy is the pseudo-Dirac pattern. The pseudo-Dirac pair can come as a sum (difference) of two sub-matrices both with maximal mixing, one hierarchal and the other inverse-hierarchal. This is clearly evident from the last column of Table I. We see the pseudo-Dirac pairs $\mathbb{C}_1 = \mathbb{A} - \mathbb{B}$ and $\mathbb{C}_2 = \mathbb{B} - \mathbb{A}$ with both \mathbb{A} and \mathbb{B} containing maximal mixing.

The decomposition of quasi-degenerate spectra can easily be incorporated within models of neutrino masses. For example, in the Type I seesaw mechanism (with two generations) the mass matrix is given by

$$\begin{aligned}
 -\mathbb{M}_\nu^I &= v^2 \begin{pmatrix} h_{ee}^D & h_{\mu e}^D \\ h_{e\mu}^D & h_{\mu\mu}^D \end{pmatrix} \begin{pmatrix} 1/M_{R1} & 0 \\ 0 & 1/M_{R2} \end{pmatrix} \begin{pmatrix} h_{ee}^D & h_{e\mu}^D \\ h_{\mu e}^D & h_{\mu\mu}^D \end{pmatrix} \\
 &= m_1 \begin{pmatrix} (h_{ee}^D)^2 & h_{ee}^D h_{e\mu}^D \\ h_{ee}^D h_{e\mu}^D & (h_{e\mu}^D)^2 \end{pmatrix} + m_2 \begin{pmatrix} (h_{\mu e}^D)^2 & h_{\mu e}^D h_{\mu\mu}^D \\ h_{\mu e}^D h_{\mu\mu}^D & (h_{\mu\mu}^D)^2 \end{pmatrix}, \tag{14}
 \end{aligned}$$

where m_1 and m_2 are given as $v^2/(M_{R1})$ and $v^2/(M_{R2})$ respectively. Each of these sub-matrices is result of a seesaw mechanism with one right-handed neutrino. Comparing the above with Eq. (9), we can determine the parameter regions required for quasi-degeneracy and large mixing. For, $M_{R1} = M_{R2}$, we see that for the Yukawa parameters, there are two choices where the mixing in the sub-matrices is small

$$\begin{aligned}
 h_{e\mu}^D &\sim \mathcal{O}(1), \quad h_{ee}^D \sim x, \quad h_{\mu\mu}^D \sim -y, \quad h_{\mu e}^D \sim \mathcal{O}(1), \quad \text{or} \\
 h_{ee}^D &\sim \mathcal{O}(1), \quad h_{e\mu}^D \sim -x, \quad h_{\mu e}^D \sim y, \quad h_{\mu\mu}^D \sim \mathcal{O}(1). \tag{15}
 \end{aligned}$$

Thus each right-handed neutrino couples with the Standard Model neutrinos with only small mixing angles, but total mass matrix ensures maximal mixing angles for the above choice of parameters. These conditions are already known in the literature for some time [14]. So this is an alternative approach of arriving at these conditions. The interesting aspect of Type I seesaw mechanism is that the decomposition at the neutrino mass matrix level can be realised at the Lagrangian level in terms of independent parameters with ‘independent mass’ scales for the the individual sub-matrices, for instance the sub-matrices have mass scales v^2/M_{R1} and v^2/M_{R2} . Such a realisation might not be possible in other models for degenerate neutrinos like in Type II seesaw mechanism. A further interesting possibility would be to consider the case when there are two independent seesaw mechanisms at work.

B. Left-Right Symmetric Model

The simplest model where the above mechanism can be realised is the LRS model. In the recent years, this model has been thoroughly analyzed for its duality properties [9]. The LRS model naturally contains both Type I and Type II seesaw contributions, which can be thought of as two sub-matrices discussed above. Further more, these models are characterized by a common Yukawa coupling to both the left-handed and right-handed Majorana mass matrices

$$\mathcal{L}_M = -\frac{f}{2} (\overline{\nu}_L^c \nu_L \Delta_L^0 + \overline{\nu}_R^c \nu_R \Delta_R^0) + h.c. , \quad (16)$$

where $\Delta_{L(R)}$ is the triplet Higgs field whose neutral component attains a vacuum expectation value (vev) giving rise to the Majorana mass to the left (right) handed neutrino fields. In addition the Dirac neutrino Yukawa coupling is also present

$$\mathcal{L}_D = -Y \overline{\nu}_L \nu_R \phi^0 + h.c. \quad (17)$$

In the limit where $v_R \gg v$, the Type I seesaw mechanism becomes operative and the total neutrino mass matrix is now given as

$$\mathbb{M}_\nu = f v_L - \frac{v^2}{v_R} Y f^{-1} Y^T . \quad (18)$$

Along the lines of the discussion we had for the two-generation case, Eq. (9), we can assume the contribution (first term on the RHS of Eq. (18)), due to Type II to be hierarchial with small mixing and second part due to the Type I contribution inverse hierarchial with small mixing. The appropriate choice of the Yukawa textures in this case are as follows

$$f = \begin{pmatrix} 0 & x \\ x & 1 \end{pmatrix} , \quad Y = \begin{pmatrix} 1 & y \\ y & 0 \end{pmatrix} . \quad (19)$$

With this choice the total mass matrix takes the form

$$\begin{aligned} \mathbb{M}_\nu &= \begin{pmatrix} 0 & m_1 x \\ m_1 x & m_1 \end{pmatrix} + \frac{m_2}{x^2} \begin{pmatrix} 1 - 2xy & y(1 - xy) \\ y(1 - xy) & y^2 \end{pmatrix} \\ &= \frac{1}{x^2} \begin{pmatrix} m_2(1 - 2xy) & m_1 x^3 + m_2 y(1 - xy) \\ m_1 x^3 + m_2 y(1 - xy) & m_1 x^2 + m_2 y^2 \end{pmatrix} , \end{aligned} \quad (20)$$

where $m_1 = v_L$ and $m_2 = v^2/v_R$. The mixing angle in the above mass matrix is given by

$$\tan 2\theta = \frac{2[m_1 x^3 + m_2 y(1 - xy)]}{[m_1 x^2 + m_2 y^2] - m_2(1 - 2xy)} . \quad (21)$$

From the above it is clear that the degeneracy requirement $m_1 x^2 \approx m_2$ automatically leads to large mixing, $\tan 2\theta \sim \mathcal{O}(\frac{1}{2y})$. A rough idea of how stable this mixing would be under radiative corrections can be obtained by considering the modification of the neutrino mass matrix below the seesaw scale. The modification is set by the matrix $\mathbb{P} = \text{Diag}\{1, 1 + \delta_\mu\}$ and is given as $\mathbb{M}_\nu = \mathbb{P}\mathbb{M}_\nu\mathbb{P}$. In this case, the mixing angle now takes the form

$$\tan 2\theta = \frac{2[m_1 x^3 + m_2 y(1 - xy)](1 + \delta_\mu)}{(1 + \delta_\mu)^2[m_1 x^2 + m_2 y^2] - m_2(1 - 2xy)}, \quad (22)$$

where $\delta_\mu = c h_\mu^2 / (16\pi^2) \log(M_X/M_W)$ specifies the size of the radiative corrections induced by the Yukawa coupling of the μ , h_μ . Here c is a constant depending on whether the theory is supersymmetric or not and M_X is the high scale just below the seesaw scale [15]. The condition for large mixing case now gets modified as $[m_1 x^2 + m_2 y^2](1 + \delta_\mu)^2 \approx m_2$. Note that this same condition is also required to keep the degeneracy stable even after radiative corrections. Of course, the splitting of the degeneracy can come from the radiative effects. A more detailed analysis of radiative corrections will be presented elsewhere.

III. EXTENSION TO THREE GENERATIONS

Let us extend the analysis of the previous section to the case of three generations. Here we will consider two cases - (a) Case I: two seesaw mechanisms or two sources of neutrino masses (Sec. III A), and (b) Case II: three seesaw mechanisms or three sources of neutrino masses (Sec. III B).

A. Case I: Two Seesaw Mechanisms

As before, let us consider two 3×3 mass matrices each with a small mixing angle and one large eigenvalue, $\mathbb{M}_\nu = \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)}$. Instead of representing them as general mass matrices as we have done for the case of two generations, we will represent them by using

$$\mathbb{M}_\nu^{(i)} = [\mathbb{U}_{\text{mix}}^{(i)}]^T \cdot \text{Diag}[\mathbb{M}_\nu^{(i)}] \cdot \mathbb{U}_{\text{mix}}^{(i)}, \quad (23)$$

where $\text{Diag}[\mathbb{M}_\nu^{(i)}] = \text{Diag}[\{m_1^{(1)}, m_2^{(2)}, m_3^{(3)}\}]$, the eigenvalues of the mass matrices and $\mathbb{U}_{\text{mix}}^{(i)}$ represents the mixing present in each of the mass matrices, with $i = 1, 2$. Given that the mixing angles in $\mathbb{M}_\nu^{(i)}$ are small, we can expand $\mathbb{U}_{\text{mix}}^{(i)}$ in terms of small parameters $\cos \theta_m^{(i)} \approx 1$, $\sin \theta_m^{(i)} \approx \epsilon_m^{(i)}$,

where $m = \{12, 23, 13\}$ labels the three angles. The total mass matrix now takes the form

$$\mathbb{M}_\nu = \begin{pmatrix} m_1^{(1)} + m_1^{(2)} & (m_2^{(1)} - m_1^{(1)})\epsilon_{12}^{(1)} + (m_2^{(2)} - m_1^{(2)})\epsilon_{12}^{(2)} & (m_3^{(1)} - m_1^{(1)})\epsilon_{13}^{(1)} + (m_3^{(2)} - m_1^{(2)})\epsilon_{13}^{(2)} \\ * & m_2^{(1)} + m_2^{(2)} & (m_3^{(1)} - m_2^{(1)})\epsilon_{23}^{(1)} + (m_3^{(2)} - m_2^{(2)})\epsilon_{23}^{(2)} \\ * & * & m_3^{(1)} + m_3^{(2)} \end{pmatrix}, \quad (24)$$

where the symmetric elements of the matrix have been represented by *. We can determine the mixing present in the total mass matrix by diagonalising the above matrix. We have

$$\mathbb{M}'_\nu = \mathbb{U}_{23}^T \mathbb{M}_\nu \mathbb{U}_{23}, \quad (25)$$

where

$$\mathbb{U}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & \sin \theta_{23} \\ 0 & -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix},$$

with

$$\tan 2\theta_{23} = 2 \frac{(m_3^{(1)} - m_2^{(1)})\epsilon_{23}^{(1)} + (m_3^{(2)} - m_2^{(2)})\epsilon_{23}^{(2)}}{m_3^{(1)} + m_3^{(2)} - m_2^{(1)} - m_2^{(2)}}. \quad (26)$$

For this mixing to be maximal the condition would be $(m_3^{(1)} - m_2^{(1)}) = -(m_3^{(2)} - m_2^{(2)})$. This condition is similar to the one we have seen earlier for the two generation case and as argued in that case, the only natural solution is to have $m_3^{(1)} = m_2^{(2)}$ with $m_2^{(1)}, m_3^{(2)}$ negligible or $m_2^{(1)} = m_3^{(2)}$ with $m_3^{(1)}, m_2^{(2)}$ negligible. We now proceed to show that if we accept either of these two solutions, it would not be possible to have one another large mixing angle in \mathbb{M}_ν , if they have to satisfy the naturalness criteria that the large eigenvalues of the individual matrices should not cancel in the total mass matrix. Defining

$$\mathbb{U}_{13} = \begin{pmatrix} \cos \theta_{13} & 0 & \sin \theta_{13} \\ 0 & 1 & 0 \\ -\sin \theta_{13} & 0 & \cos \theta_{13} \end{pmatrix}, \quad (27)$$

we have

$$\mathbb{M}''_\nu = \mathbb{U}_{13}^T \mathbb{M}'_\nu \mathbb{U}_{13}. \quad (28)$$

$\tan 2\theta_{23}$ in the limit where the solution for maximal mixing of the 23 angle, $m_3^{(1)} = m_2^{(2)} = \bar{m}$ with $m_2^{(1)}, m_3^{(2)} \sim 0$ is taken is given by

$$\tan 2\theta_{13} \approx \frac{m_1^{(1)}(\epsilon_{12}^{(1)} + \epsilon_{13}^{(1)}) - \bar{m}(\epsilon_{13}^{(1)} + \epsilon_{12}^{(2)}) + m_1^{(2)}(\epsilon_{12}^{(2)} + \epsilon_{13}^{(2)})}{\sqrt{2}(m_1^{(1)} + m_1^{(2)} - \bar{m}(1 + \epsilon_{23}^{(1)} - \epsilon_{23}^{(2)}))}. \quad (29)$$

⁵ More precisely, we should have $m_3^{(1)} - m_2^{(2)} \approx \mathcal{O}(m_3^{(1)}(\epsilon_{23}^{(1)} - \epsilon_{23}^{(2)}))$ and $m_2^{(1)}, m_3^{(2)}$ much smaller compared to them.

From the above we realize the following conditions for (a) small mixing : $|m_1^{(1)} + m_1^{(2)} - \bar{m}| \gg 0$, and (b) maximal mixing : $|m_1^{(1)} + m_1^{(2)} - \bar{m}| = 0$. Finally, defining

$$\mathbb{U}_{12} = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\sin \theta_{12} & \cos \theta_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (30)$$

we have

$$\mathbb{M}'_{\nu} = \mathbb{U}_{12}^T \mathbb{M}''_{\nu} \mathbb{U}_{12}. \quad (31)$$

$\tan 2\theta_{12}$ has the following form in the limiting case when θ_{13} is very small

$$\tan 2\theta_{12} \approx \frac{m_1^{(1)}(\epsilon_{12}^{(1)} - \epsilon_{13}^{(1)}) + \bar{m}(\epsilon_{13}^{(1)} - \epsilon_{12}^{(2)}) + m_1^{(1)}(\epsilon_{12}^{(2)} - \epsilon_{13}^{(2)})}{\sqrt{2}(m_1^{(1)} + m_1^{(2)} - \bar{m}(1 - \epsilon_{23}^{(1)} + \epsilon_{23}^{(2)}))} + \mathcal{O}(\theta_{13}). \quad (32)$$

From the above we see that the conditions for the mixing are the same for both θ_{12} and θ_{13} in this limit. Thus either both become maximal/large or both remain small. Finally, in the limit of maximal θ_{13} mixing, the expression for $\tan 2\theta_{12}$ becomes

$$\frac{(m_1^{(1)}(\epsilon_{12}^{(1)} - \epsilon_{13}^{(1)}) + \bar{m}(\epsilon_{13}^{(1)} - \epsilon_{12}^{(2)}) + m_1^{(2)}(\epsilon_{12}^{(2)} - \epsilon_{13}^{(2)}))}{m_1^{(1)} + m_1^{(2)} + \bar{m}(-1 + 3\epsilon_{23}^{(1)} - 3\epsilon_{23}^{(2)}) + \sqrt{2}(m_1^{(1)}(\epsilon_{12}^{(1)} + \epsilon_{13}^{(1)}) - \bar{m}(\epsilon_{13}^{(1)} + \epsilon_{12}^{(2)}) + m_1^{(2)}(\epsilon_{12}^{(2)} + \epsilon_{13}^{(2)}))}, \quad (33)$$

which is also automatically maximal/large within the small $\epsilon_{ij}^{(k)}$ limit. Before we proceed, a few comments are in order regarding the ordering of the eigenvalues. In the case where there is only one maximal/large mixing, the sub matrices can have hierarchal and inverse-hierarchal patterns, with the hierarchal sub-matrix containing one large eigenvalue and the inverse-hierarchal containing two large eigenvalues. The only condition is on their CP parties; the eigenvalues taking part in the enhancement of the mixing should have the same CP parities. The list of possible forms the sub-matrices can take is discussed in the subsection III A 2 where decomposition of the degenerate spectrum is considered in three generation case.

On the other hand, for the case with all the three large/maximal mixing case, as per our arguments earlier, *i.e.*, the large eigenvalues of the individual matrices should not cancel in the total matrix, the present solution necessarily favours an *alternating* pattern for the eigenvalues for the individual mass matrices ⁶

$$\begin{aligned} \text{Diag}[\mathbb{M}_{\nu}^{(1)}] &= \text{Diag}[\{m_1^{(1)}, 0, m_3^{(1)}\}] , \quad \text{Diag}[\mathbb{M}_{\nu}^{(2)}] = \text{Diag}[\{0, m_2^{(2)}, 0\}] \\ \text{Diag}[\mathbb{M}_{\nu}^{(1)}] &= \text{Diag}[\{0, m_2^{(1)}, 0\}] , \quad \text{Diag}[\mathbb{M}_{\nu}^{(2)}] = \text{Diag}[\{m_1^{(2)}, 0, m_3^{(2)}\}] . \end{aligned} \quad (34)$$

⁶ The zeroth order textures for alternating pattern of neutrino mass matrices are given in Table IV.

In this case, the mixing pattern corresponds to the *truly maximal mixing* matrix of Cabibbo and Wolfenstein [16] along with the degeneracy condition $m_1 \approx m_2 \approx m_3$. In the recent years, the truly maximal mixing matrix has been achieved from A_4 symmetry by Ma and Rajasekaran also for degenerate case [13]. As it stands this matrix is not phenomenologically viable as all the three mixing angles it predicts are large. However there could be other corrections to this mass matrix depending on the model which would rectify this situation [17] and make the mass matrix phenomenologically viable.

1. Two Equivalent Textures

From our arguments above, it appears that we can generate only one large mixing angle in the case when there are only two sub-matrices, because of the important constraint that the third mixing angle (θ_{13}) must not be large ⁷. Given that we can only generate one large mixing from the small mixing using the degenerate conditions, we will have to assume that at least one of the sub-matrices has intrinsically one maximal/large mixing angle. However, the presence of this mixing should not disturb the smallness of θ_{13} angle in the total mass matrix. In the following, we will consider one of the sub-matrices to have pseudo-Dirac structure and other one to have one large eigenvalue and all the three mixing angles small. This is because the pseudo-Dirac structure not only gives maximal mixing but also has the eigenvalues with opposite CP parities.

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} x^2 & x & y^2 \\ x & 0 & 1 \\ y^2 & 1 & 0 \end{pmatrix} + m_2 \begin{pmatrix} 1 & z & t^3 \\ z & z^3 & t^3 \\ t^3 & t^3 & z^3 \end{pmatrix}, \quad (35)$$

where x, y, z, t are small entries compared to m_1, m_2 . We will diagonalise this matrix in the following manner. Rotating by \mathbb{O}_{23} on both sides, we have

$$\mathbb{O}_{23}^T \mathbb{M}_\nu \mathbb{O}_{23} = \begin{pmatrix} m_2 + m_1 x^2 & m_1 x \cos \theta_{23} + \tilde{m}_{12} & -m_1 x \sin \theta_{23} + \tilde{m}_{13} \\ m_1 x \cos \theta_{23} + \tilde{m}_{12} & m_1 \sin 2\theta_{23} + \tilde{m}_{22} & 0 \\ -m_1 x \sin \theta_{23} + \tilde{m}_{13} & 0 & -m_1 \sin 2\theta_{23} + \tilde{m}_{33} \end{pmatrix}, \quad (36)$$

where \mathbb{O}_{23} is defined as

$$\mathbb{O}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & -\sin \theta_{23} \\ 0 & \sin \theta_{23} & \cos \theta_{23} \end{pmatrix}, \quad (37)$$

⁷ This would be case in models where there are no large radiative corrections effecting the mixing angles strongly.

with the angle θ_{23} given by

$$\theta_{23} = \frac{1}{2} \tan^{-1} \left[\frac{2(m_1 + m_2 t^3)}{(m_2 z^3 - m_2 z^3)} \right] = \frac{\pi}{4}. \quad (38)$$

The explicit forms for \tilde{m}_{ij} can be easily deduced. A crucial point to note is that the diagonal elements of the matrix in Eq. (36) carry opposite sign for the dominant element (m_1). This would have the consequence of keeping the 13 mixing small, while making 23 mixing large, when the degeneracy condition $m_1 \approx m_2 \approx m$ is imposed. The total mixing matrix is given by $\mathbb{O} = \mathbb{O}_{12}\mathbb{O}_{13}\mathbb{O}_{23}$ with the angles θ_{13} and θ_{12} defined as

$$\begin{aligned} \theta_{13} &= \frac{1}{2} \tan^{-1} \left[\frac{2(-m_1 x \sin \theta_{23} + \tilde{m}_{13})}{-m_1 \sin 2\theta_{23} + \tilde{m}_{33} - m_1 x^2 - m_2} \right], \\ \theta_{12} &= \frac{1}{2} \tan^{-1} \left[\frac{2\tilde{m}'_{12}}{\tilde{m}'_{22} - \tilde{m}'_{11}} \right], \end{aligned} \quad (39)$$

where the explicit form of \tilde{m}'_{ij} can easily be deduced. From the above, we can see that the degeneracy induced large mixing mechanism works for the 12 mixing, while it does not generate large (maximal) mixing for the 13 mixing angle. This is due to the choice of having $\tau\tau$ element with opposite sign (loosely speaking CP parity) compared to the $\mu\mu$ element.

The above Yukawa matrices can be easily incorporated in the LRS model by choosing f and Y of Eq. (18) (at the leading order) as

$$f = \begin{pmatrix} x^2 & x & y^2 \\ x & 0 & 1 \\ y^2 & 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (40)$$

Notice that it reproduces the Eq. (35) at the zeroth order. From the discussion in the previous section, we also know that the pseudo-Dirac mass matrix can be decomposed in to maximally mixing sub-matrices. Thus another texture which could equally give the same results is given by

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} x^2 & x & y^2 \\ x & \frac{1}{2} & \frac{1}{2} \\ y^2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + m_2 \begin{pmatrix} 1 & z & t^3 \\ z & \frac{1}{2} & -\frac{1}{2} \\ t^3 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (41)$$

The first of the matrices has only one large eigenvalue in a hierarchial pattern with maximal mixing, whereas the second one has two large eigenvalues with one maximal mixing and two small mixings with inverted hierarchy. Lets emphasize once more that one needs opposite eigenvalues $m_1 \approx -m_2$ to obtain the large atmospheric mixing in this case.

2. Decomposition of the Degenerate Spectrum

For three generations the decomposition of the degenerate spectrum into hierarchical and inverse-hierarchical mass patterns is straight forward. In Table II, we present the zeroth order mass matrices for the three generation case. Note that the present notation has been previously used in the

Mixing \Rightarrow \mathbb{M}_{diag}	Small \mathbb{X}_ϵ	Single maximal \mathbb{X}_{SM}	Bimaximal \mathbb{X}_{BM}	Tribimaximal \mathbb{X}_{TBM}
Hierarchical A: Diag[0,0,1]	$\begin{pmatrix} 0 & 0 & \epsilon_{13} \\ 0 & 0 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
Inverse hierarchical \mathbb{B}_1 : Diag[1,-1,0] \mathbb{B}_2 : Diag[1,1,0]	$\begin{pmatrix} 1 & -2\epsilon_{12} & -\epsilon_{13} \\ -2\epsilon_{12} & -1 & \epsilon_{23} \\ -\epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & -\epsilon_{13} \\ 0 & 1 & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ \frac{2}{3} & \frac{1}{6} & -\frac{1}{6} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$
Degenerate \mathbb{C}_0 : Diag[1,1,1] \mathbb{C}_1 : Diag[-1,1,1] \mathbb{C}_2 : Diag[1,-1,1] \mathbb{C}_3 : Diag[1,1,-1]	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & 2\epsilon_{12} & 2\epsilon_{13} \\ 2\epsilon_{12} & 1 & 0 \\ 2\epsilon_{13} & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & -2\epsilon_{12} & 0 \\ -2\epsilon_{12} & -1 & 2\epsilon_{23} \\ 0 & 2\epsilon_{23} & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & -2\epsilon_{13} \\ 0 & 1 & -2\epsilon_{23} \\ -2\epsilon_{13} & -2\epsilon_{23} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

TABLE II: Different standard textures (zeroth order) for different combinations of mixings (setting m_1 , m_2 and m_3 as dimensionless quantities which are either zero or one depending on the different cases listed) consistent with data ^a.

^aIn Ref. 14, only two cases (single and bimaximal mixing) were considered and they used $\mathbb{M}_\nu = \mathbb{U}_{PMNS}^\dagger \mathbb{M}_{diag} \mathbb{U}_{PMNS}$, which is different from our definition (Eq. (13)).

literature [14] and all the matrices present in this table have been used **previously** to describe the neutrino mass matrix at the zeroth order. After adding *small* perturbations to these matrices they can explain the neutrino data. However, as before we are interested in only decomposing the degenerate mass matrix in terms of the hierarchical and inverse-hierarchical mass matrices. As before, from each of the columns, we can see that each degenerate case can be constructed as a

sum of hierarchal and inverse hierarchal textures. For example, \mathbb{C}_0 can be considered as $\mathbb{A} + \mathbb{B}_2$. Similarly, \mathbb{C}_1 can be considered as $-\mathbb{B}_1 + \mathbb{A}$ and so on. And this is true as we go along each of the columns, *i.e* for all kinds of mixing angles. This simple observation can be restated as *every degenerate neutrino mass matrix can be thought of a sum of hierarchal and inverse hierarchal sub-mass matrices while the converse is not generally true*. In three generations, the above set

Mixing \Rightarrow M_{diag}	Small \mathbb{X}_ϵ	Single maximal \mathbb{X}_{SM}	Bimaximal \mathbb{X}_{BM}	Tribimaximal \mathbb{X}_{TBM}
$\tilde{\mathbb{A}}_1: \text{Diag}[0,1,1]$	$\begin{pmatrix} 0 & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & 1 & 0 \\ \epsilon_{13} & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{5}{6} \end{pmatrix}$
$\tilde{\mathbb{A}}_2: \text{Diag}[0,1,-1]$	$\begin{pmatrix} 0 & \epsilon_{12} & -\epsilon_{13} \\ \epsilon_{12} & 1 & -2\epsilon_{23} \\ -\epsilon_{13} & -2\epsilon_{23} & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{3}{4} & -\frac{1}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{6} & -\frac{5}{6} \\ -\frac{1}{3} & -\frac{5}{6} & -\frac{1}{6} \end{pmatrix}$
$\tilde{\mathbb{B}}: \text{Diag}[1,0,0]$	$\begin{pmatrix} 1 & -\epsilon_{12} & -\epsilon_{13} \\ -\epsilon_{12} & 0 & 0 \\ -\epsilon_{13} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}$

TABLE III: Novel textures (leading order) for different mixing scenarios which by themselves need not be consistent with data. These cases are useful when we consider adding two different textures to obtain the degenerate cases. The labels with tilde sign are new textures by taking into account the fact that hierarchy or inverse hierarchy can appear in either 1-2 sector or the 2-3 sector respectively. The standard textures considered degeneracy in 1-2 sector and hierarchy or inverse hierarchy only in the 2-3 sector.

of decomposition which is based on neutrino data is not exhaustive. This is essentially because the constraints of the neutrino data are not on the individual sub-matrices but on the total mass matrix. In such a case, the normal and inverse hierarchal sub matrices can take other possible forms \mathbb{A} and \mathbb{B}_i than those listed in Table II. From Table III, it is easy to see that the combinations of $\tilde{\mathbb{A}}_i$ and $\tilde{\mathbb{B}}$ would produce one of the degenerate textures \mathbb{C}_i of the original Table II. However, even this list is not exhaustive for the degenerate case. We could have textures which are not traditionally ordered as either hierarchal or inverse hierarchal in the three generation case. These cases are listed in Table IV and we call them as alternating textures (see Eq. (34)). Thus in summary, we have covered all possible ways of ordering the three degenerate eigenvalues in to two sub-matrices, which are not degenerate themselves.

Mixing \Rightarrow \mathbb{M}_{diag}	Small \mathbb{X}_ϵ	Single maximal \mathbb{X}_{SM}	Bimaximal \mathbb{X}_{BM}	Tribimaximal \mathbb{X}_{TBM}
\mathbb{T}_1 : $\text{Diag}[0,1,0]$	$\begin{pmatrix} 0 & \epsilon_{12} & 0 \\ \epsilon_{12} & 1 & -\epsilon_{23} \\ \epsilon_{13} & -\epsilon_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$
\mathbb{T}_2 : $\text{Diag}[1,0,1]$	$\begin{pmatrix} 1 & -\epsilon_{12} & 0 \\ -\epsilon_{12} & 0 & \epsilon_{23} \\ 0 & \epsilon_{23} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$

TABLE IV: Alternating textures (leading order) for different mixing scenarios.

B. Case II: Three Sources

For more than two seesaw mechanisms at work, the generalisation is straight forward. Lets consider the case where there are three sources of neutrino masses. The total mass matrix in this case is given by

$$\mathbb{M}_\nu = \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)} + \mathbb{M}_\nu^{(3)}, \quad (42)$$

where each of the sub matrices can be thought of having independent origin through a seesaw mechanism or any other scheme to generate non-zero neutrino masses. As with the two-generation case, we will now consider the case where all the mixings present in each of the sub matrices are taken to be small and each sub-matrix is assumed to have only one large eigenvalue. The second assumption is a direct consequence of assuming that all the three sources contribute equally and there are no cancellations between the dominant eigenvalues of the sub-matrices. With these assumptions, the total mass matrix can now be written in terms of the individual mass matrices as

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} \epsilon_{13}^2 & \epsilon_{13}\epsilon_{23} & \epsilon_{13} \\ \epsilon_{13}\epsilon_{23} & \epsilon_{23}^2 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 1 \end{pmatrix} + m_2 \begin{pmatrix} \epsilon'_{12} & \epsilon'_{12} & \epsilon'_{12}\epsilon'_{23} \\ \epsilon'_{12} & 1 & -\epsilon'_{23} \\ \epsilon'_{12}\epsilon'_{23} & -\epsilon'_{23} & \epsilon'_{23}^2 \end{pmatrix} + m_3 \begin{pmatrix} 1 & -\epsilon''_{12} & -\epsilon''_{13} \\ -\epsilon''_{12} & \epsilon''_{12} & \epsilon''_{12}\epsilon''_{13} \\ -\epsilon''_{13} & \epsilon''_{12}\epsilon''_{13} & \epsilon''_{13}^2 \end{pmatrix}, \quad (43)$$

where $\epsilon_{ij}, \epsilon'_{ij}, \epsilon''_{ij}$ ($i, j = 1, 2, 3$) are small entries corresponding to small mixing angles in $U_{mix}^{(i)}$. This total matrix can be diagonalised by an orthogonal matrix $\mathbb{O} \equiv \mathbb{O}_{23}\mathbb{O}_{13}\mathbb{O}_{12}$, such that $\mathbb{O}^T \mathbb{M}_\nu \mathbb{O} = \text{Diag}[\mathbb{M}_\nu]$. \mathbb{O}_{ij} represents a rotation in the ij^{th} plane. For example

$$\mathbb{O}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_{23} & \sin \theta_{23} \\ 0 & -\sin \theta_{23} & \cos \theta_{23} \end{pmatrix}. \quad (44)$$

$$\begin{aligned}
\theta_{23} &\approx \frac{1}{2} \tan^{-1} \left[\frac{2(m_1 \epsilon_{23} - m_2 \epsilon'_{23} + m_3 \epsilon''_{12} \epsilon''_{13})}{m_1(1 - \epsilon_{23}^2) - m_2(1 - \epsilon'_{23}{}^2) + m_3(\epsilon''_{13}{}^2 - \epsilon''_{12}{}^2)} \right], \\
\theta_{13} &\approx \frac{1}{2} \tan^{-1} \left[\frac{2(\tilde{m}_{13})}{\tilde{m}_{33} - \tilde{m}_{11}} \right], \\
\theta_{12} &\approx \frac{1}{2} \tan^{-1} \left[\frac{2(\tilde{m}'_{12})}{\tilde{m}'_{22} - \tilde{m}'_{11}} \right],
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
\tilde{m}_{13} &= s_{23}(m_1 \epsilon_{13} \epsilon_{23} + m_2 \epsilon'_{12} - m_3 \epsilon''_{12}) + c_{23}(m_1 \epsilon_{13} + m_2 \epsilon'_{12} \epsilon'_{13} - m_3 \epsilon''_{13}), \\
\tilde{m}_{33} &= s_{23}[s_{23}(m_2 + m_1 \epsilon_{23}^2 + m_3 \epsilon''_{12} \epsilon''_{13}) + c_{23}(m_1 \epsilon_{23} - m_2 \epsilon'_{23} + m_3 \epsilon''_{12} \epsilon''_{13})] \\
&\quad + c_{23}[s_{23}(m_1 \epsilon_{23} - m_2 \epsilon'_{23} + m_3 \epsilon''_{12} \epsilon''_{13}) + c_{23}(m_1 + m_2 \epsilon_{23}^2 + m_3 \epsilon''_{13}{}^2)], \\
\tilde{m}_{11} &= m_3 + m_1 \epsilon_{13}^2 + m_2 \epsilon'_{12}{}^2, \\
\tilde{m}'_{12} &= c_{13} \tilde{m}_{12} = c_{13}[c_{23}(m_1 \epsilon_{13} \epsilon_{23} + m_2 \epsilon'_{12} - m_3 \epsilon''_{12}) - s_{23}(m_1 \epsilon_{13} + m_2 \epsilon'_{12} \epsilon'_{13} - m_3 \epsilon''_{13})], \\
\tilde{m}'_{22} &= \tilde{m}_{22} = c_{23}[c_{23}(m_2 + m_1 \epsilon_{23}^2 + m_3 \epsilon''_{12} \epsilon''_{13}) - s_{23}(m_1 \epsilon_{23} - m_2 \epsilon'_{23} + m_3 \epsilon''_{12} \epsilon''_{13})] \\
&\quad - s_{23}[c_{23}(m_1 \epsilon_{23} - m_2 \epsilon'_{23} + m_3 \epsilon''_{12} \epsilon''_{13}) - s_{23}(m_1 + m_2 \epsilon_{23}^2 + m_3 \epsilon''_{13}{}^2)], \\
\tilde{m}'_{11} &= c_{13}(\tilde{m}_{11} c_{13} - \tilde{m}_{13} s_{13}) - s_{13}(\tilde{m}_{13} c_{13} - \tilde{m}_{33} s_{13}).
\end{aligned} \tag{46}$$

Notice that all the three mass eigenvalues are of the same CP parity in the above and the degeneracy induced mixing thus works for the all the three mixing angles. Thus all the three mixing angles are large. One can then ask the question whether choosing one of the mass eigenvalues with a negative CP parity would help in keeping one of the mixing angles small. The answer is negative, choosing one of the eigenvalues to have CP parity negative leads to at least two of the mixing angles to remain small as the degeneracy induced large mixing mechanism is no longer operative for two of the mixing angles. Thus we are back to the case of two seesaw mechanisms which we have seen in the previous subsection.

While it is possible to visualise GUT models where there are three seesaw mechanisms at work, it much easier to suitably split a single Type I seesaw mass matrix into three sub-matrices. In this case, we can extend Eq. (14) to three generations as

$$\begin{aligned}
-\mathbb{M}_\nu^I &= m_1 \begin{pmatrix} (h_{ee}^D)^2 & h_{ee}^D h_{e\mu}^D & h_{ee}^D h_{e\tau}^D \\ h_{ee}^D h_{e\mu}^D & (h_{e\mu}^D)^2 & h_{e\mu}^D h_{e\tau}^D \\ h_{ee}^D h_{e\tau}^D & h_{e\mu}^D h_{e\tau}^D & (h_{e\tau}^D)^2 \end{pmatrix} + m_2 \begin{pmatrix} (h_{\mu e})^2 & h_{\mu e}^D h_{\mu\mu}^D & h_{\mu e}^D h_{\mu\tau}^D \\ h_{\mu e}^D h_{\mu\mu}^D & (h_{\mu\mu}^D)^2 & h_{\mu\mu}^D h_{\mu\tau}^D \\ h_{\mu e}^D h_{\mu\tau}^D & h_{\mu\mu}^D h_{\mu\tau}^D & (h_{\mu\tau}^D)^2 \end{pmatrix} \\
&\quad + m_3 \begin{pmatrix} (h_{\tau e})^2 & h_{\tau e}^D h_{\tau\mu}^D & h_{\tau e}^D h_{\tau\tau}^D \\ h_{\tau e}^D h_{\tau\mu}^D & (h_{\tau\mu}^D)^2 & h_{\tau\mu}^D h_{\tau\tau}^D \\ h_{\tau e}^D h_{\tau\tau}^D & h_{\tau\mu}^D h_{\tau\tau}^D & (h_{\tau\tau}^D)^2 \end{pmatrix}.
\end{aligned} \tag{47}$$

Comparing this with Eq. (43), we see that we will have three possible solutions for the Yukawa couplings in this case. The first solution is

$$\begin{aligned}
h_{ee}^D &\sim \epsilon_{13} , \quad h_{e\mu}^D \sim \epsilon_{23} , \quad h_{e\tau}^D \sim \mathcal{O}(1) , \\
h_{\mu e}^D &\sim \epsilon'_{12} , \quad h_{\mu\mu}^D \sim \mathcal{O}(1) , \quad h_{\mu\tau}^D \sim -\epsilon'_{23} , \\
h_{\tau e}^D &\sim \mathcal{O}(1) , \quad h_{\tau\mu}^D \sim -\epsilon''_{12} , \quad h_{\tau\tau}^D \sim -\epsilon''_{13} .
\end{aligned} \tag{48}$$

There are two more possibilities given by

$$\begin{aligned}
h_{\mu e}^D &\sim \epsilon_{13} , \quad h_{\mu\mu}^D \sim \epsilon_{23} , \quad h_{\mu\tau}^D \sim \mathcal{O}(1) , \\
h_{ee}^D &\sim \epsilon'_{12} , \quad h_{e\mu}^D \sim \mathcal{O}(1) , \quad h_{e\tau}^D \sim -\epsilon'_{23} , \\
h_{\tau e}^D &\sim \mathcal{O}(1) , \quad h_{\tau\mu}^D \sim -\epsilon''_{12} , \quad h_{\tau\tau}^D \sim -\epsilon''_{13} ,
\end{aligned} \tag{49}$$

or

$$\begin{aligned}
h_{\tau e}^D &\sim \epsilon_{13} , \quad h_{\tau\mu}^D \sim \epsilon_{23} , \quad h_{\tau\tau}^D \sim \mathcal{O}(1) , \\
h_{\mu e}^D &\sim \epsilon'_{12} , \quad h_{\mu\mu}^D \sim \mathcal{O}(1) , \quad h_{\mu\tau}^D \sim -\epsilon'_{23} , \\
h_{ee}^D &\sim \mathcal{O}(1) , \quad h_{e\mu}^D \sim -\epsilon''_{12} , \quad h_{e\tau}^D \sim -\epsilon''_{13} .
\end{aligned} \tag{50}$$

From the above we see that even if each of the leptonic generation couples *minimally* with each of the right-handed neutrino, the total mixing can be maximal, purely due to the degeneracy requirement. The comments at the end of subsection III A regarding maximally symmetric leptonic mixing matrix hold in this case too. Finally, note that each set of these solutions is related by S_3 symmetry to the other set.

IV. SUMMARY

In the present work, we have concentrated on the case with two seesaw mechanisms at work which occurs naturally in many examples like LRS models, SO(10) based GUT models *etc.* We have shown that if both these seesaw mechanisms result in mass matrices which only have small mixing in them, then the only pattern of mass eigenvalues which is *naturally* consistent with maximal/large mixing is the quasi-degenerate pattern for the total mass matrix.

All the arguments presented in the present work are independent of the details of the sources of neutrino masses. However, depending on the specifics of the model, there could be radiative corrections which could significantly modify the mixing angles. For example, if one has Type I + Type II seesaw mechanism operating at the high scale, radiative corrections could significantly

modify the mixing angles at the weak scale. These effects should be taken in to account when applying the results of the present work to any particular model. The impact of radiative corrections, models and implications for leptogenesis within this class of *hybrid degenerate models* are being studied for a future publication [18].

APPENDIX A: GENERALIZATION OF THE RESULT FOR n SOURCES

If there are n sources of neutrino masses in a particular model such that the total mass matrix is given by

$$\mathbb{M}_\nu = \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)} + \dots + \mathbb{M}_\nu^{(n)}. \quad (\text{A1})$$

And further each of the $\mathbb{M}_\nu^{(i)}$ have one dominant diagonal element proportional to its largest eigenvalue m_i , and rest of the entries to be tiny (all the mixing angles in all the $\mathbb{M}_\nu^{(i)}$ are small); $\mathbb{M}_\nu^{(i)}$ are ordered in such a way that the ii^{th} element is dominant. There are n possible orderings of $\mathbb{M}_\nu^{(i)}$. Then the total mass matrix would *naturally* have a quasi-degenerate pattern with maximal/large mixing depending on the number of *pairs* of eigenvalues which have the same CP parity, if $m_1 \approx m_2 \approx \dots \approx m_n$. If there are l eigenvalues with the same CP parity ⁸, then ${}^n\mathcal{C}_2 + {}^{n-l}\mathcal{C}_2$ (if $(n-l) > 2$) angles will be large or maximal and the remaining will be small. An important exception to the above is the pseudo-Dirac pattern of degenerate masses, which can only result from a ‘sum’ of two mass matrices both containing maximal mixing and equal eigenvalues with opposite ordering in hierarchy. Conversely, at the zeroth order a $n \times n$ quasi-degenerate matrix with eigenvalues $m_1, m_2, \dots, m_i \dots m_n$ (by definition $m_1 \approx m_2 \approx \dots \approx m_n$) can be decomposed in to n sub-matrices $\mathbb{M}_\nu^{(n)}$, with eigenvalues distributed as

$$\begin{aligned} \mathbb{M}_\nu &= \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)} + \dots + \mathbb{M}_\nu^{(n)} \\ \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix} &= \begin{pmatrix} m_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}. \end{aligned} \quad (\text{A2})$$

This holds true irrespective of the mixing present in the total mass matrix \mathbb{M}_ν .

⁸ And if the splitting between relevant m_i is smaller than the tiny off-diagonal entries.

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