Topological phase in two flavor neutrino oscillations

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Abstract

We show that the phase appearing in neutrino flavor oscillation formulae has a geometric and topological contribution. We identify a topological phase appearing in the two flavor neutrino oscillation formula using Pancharatnam's prescription of quantum collapses between nonorthogonal states. Such quantum collapses appear naturally in the expression for appearance and survival probabilities of neutrinos. Our analysis applies to neutrinos propagating in vacuum or through matter. For the minimal case of two flavors with CP conservation, our study shows for the first time that there is a geometric interpretation of the neutrino oscillation formulae for the detection probability of neutrino species.

Introduction

Neutrino oscillations

- Pontecorvo proposed $\bar{\nu} \to \nu$ (similar to $K_0 \to \bar{K}_0$) oscillations in 1957 to explain the rumors that neutrinos were observed in β-decay experiment of Davis (only one neutrino known at that time !).
- The rumors turned out to be just so but the remarkable insight remained, and today Neutrino flavor oscillations among the three flavors of light active neutrinos (conserving L) is the preferred solution to the anomalies in the neutrino sector in a wide variety of systems (sun, atmospheric, reactor, accelerator).
- In Standard Model, neutrinos are strictly massless $\rightsquigarrow$ can not oscillate and therefore neutrino flavor oscillations provide the only firm evidence in favor of physics beyond the Standard Model of particle physics, even though there are other strong indications (Baryon Asymmetry of the Universe, Dark Matter, Dark Energy, etc.)

Why do neutrinos oscillate ?

- Neutrinos are produced and detected via weak interaction : Weak (flavor) eigenstates differ from the stationary (mass) states of the Hamiltonian, infact they are linear superpositions of the stationary mass states.
- This leads to oscillation phenomena which is very similar to birefringence in optics - depends on properties of the medium.
- Oscillations of neutrinos takes place even in vacuum. This is driven by non-zero mass-squared splittings and non-zero mixing angles.
- In matter, oscillations are still driven by the non-zero mass-squared splittings and non-zero mixing angles which get modified due to charged current potential for $\nu_e - e$ coherent forward scattering.
- Incoherent scattering cross-section is usually negligible and this leads to sustained coherence over astrophysical length scales.

Dispersion relation for two flavor neutrinos

In Vacuum

Kim and Pevsner

$$E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \qquad \mathbb{H}_\nu = \left(p + \frac{m_1^2 + m_2^2}{4p}\right)\mathbb{I} + \frac{1}{2}\begin{pmatrix} -\omega\cos 2\theta & \omega \sin 2\theta \\ \omega\sin 2\theta & \omega\cos 2\theta \end{pmatrix} \quad \text{no parameter}$$

- $\omega = \frac{\delta m^2}{2p} = \frac{(m_i^2 - m_j^2)}{2p}$ is the vacuum frequency and θ is mixing angle in vacuum
- Assume equal and fixed momenta, $p_1 = p_2 = p$ (monochromatic). Two flavor neutrinos $\Rightarrow$ Two state quantum system
- Hilbert space of this system can be mapped onto a Blöch sphere (analogous to Poincaré sphere in optics)

In Ordinary Medium + Standard interactions

Kim and Pevsner

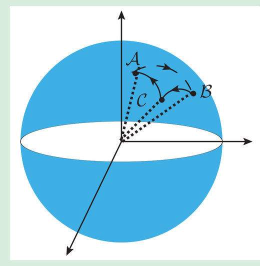
$$\mathbb{H}_\nu = \left(p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_C}{2} + V_N\right)\mathbb{I} + \frac{1}{2}\begin{pmatrix} V_C - \omega\cos 2\theta & \omega \sin 2\theta \\ \omega\sin 2\theta & -(V_C - \omega\cos 2\theta) \end{pmatrix} \quad \text{one parameter}$$

- $V_C = \sqrt{2}G_F n_e$ and $V_N = -\sqrt{2}G_F n_n/2$ are the induced potentials due to neutrino matter (e, n, p) coherent forward scattering, n_f is the number density of fermions ($f = e, n, p$)
- Most dramatic effect is the Mikhevev-Smirnov-Wolfenstein (MSW) resonance due to vanishing of the diagonal terms Wolfenstein (1978), Mikhevev and Smirnov (1985)
- Absence of flavor changing neutral currents $\Rightarrow$ if vacuum mixing and mass-splitting is zero, then matter does not really matter (in oscillations).

The Pancharatnam phase

in quantum mechanical language

Pancharatnam (1956), Berry, (1987), Samuel and Bhandari (1988)



- Notion of geometric parallelism from inner product of two states
- Reference condition is the Pancharatnam's connection $\langle A | B \rangle$ is real and positive, in phase or parallel $|| |A\rangle + |B\rangle ||^2 = \langle A|A\rangle + \langle B|B\rangle + 2|\langle A|B\rangle| \cos\{ph\langle A|B\rangle\}$
- Geometrically, norm of the resultant vector is maximum
- Physically, interference of superposed beams gives maximum intensity/probability
- The connection is both symmetric and reflexive but not transitive and this fact leads to
- Pancharatnam's phase given by phase of the complex number $\langle A|C\rangle\langle C|B\rangle\langle B|A\rangle \equiv re^{i\beta}$
- β reflects curvature of the projective Hilbert space. $\beta = \frac{\Omega}{2}$ (Ω is solid angle subtended by geodesic triangle $\mathcal{ABC}$ at origin)
- Essential requirements - minimum 3 states for non-transitivity and exploring the curvature of the ray space and cyclic projections, the state need not be an eigenstate of $\mathbb{H}$.

Collapse process

- Schrödinger evolution possibly interrupted by measurements can lead to Pancharatnam's phase. If we take any state and subject it to multiple quantum collapses and bring it back to itself, the resulting state is

$$|A\rangle\langle A|C\rangle\langle C|B\rangle\langle B|A\rangle$$

where the phase of the complex number is half the solid angle Ω subtended by the geodesic polygon at the center of the sphere.

- In order to detect geometric phases, the key ingredient is a SPLIT BEAM EXPERIMENT

$$|| |\psi_1\rangle + e^{i\beta}|\psi_2\rangle ||^2 = \langle\psi_1|\psi_1\rangle + \langle\psi_2|\psi_2\rangle + e^{i\beta}\langle\psi_1|\psi_2\rangle + e^{-i\beta}\langle\psi_2|\psi_1\rangle$$

Split beam experiment with neutrinos ?

- Incoherent scatterings are small in most practical situations (oscillation length being much smaller than mean free path in medium).

$$l_{osc} = \frac{\pi}{1.27}10^2 cm \frac{E}{[MeV]}\frac{[eV^2]}{\delta m^2} \quad << \quad l_{mfp} = 10^{19} cm \frac{[g\ cm^{-3}]}{\rho}\frac{[MeV]}{E} = 10\ light\ years$$

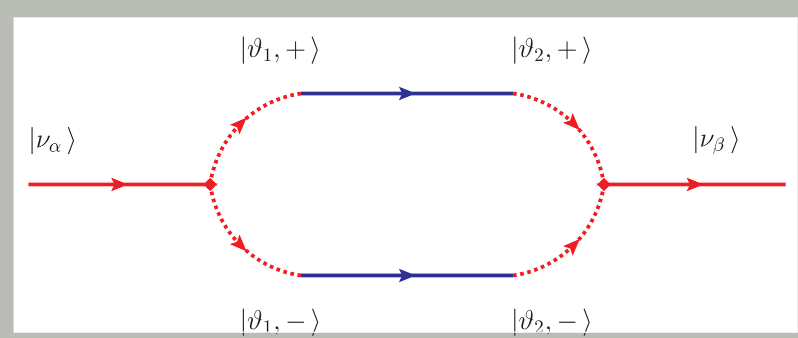
Coherence maintained over astrophysical scales.

- At zero temperature, total refractive index is

$$n_{refr} - 1 = -\frac{(\sqrt{2}G_F n_e)}{E} + i\frac{1}{2E}n_e\sigma_T \simeq -10^{-19}\frac{\rho}{[g\ cm^{-3}]}\frac{[MeV]}{E} + \dots$$

Mostly $\Re e[n_{refr}] >> \Im m[n_{refr}]$. Its potentially observable effect occurs in neutrino oscillations which probes effects due to small mass-splittings.

- With neutrinos it is not possible to design a split beam interference experiment owing to their weakly interacting nature.
- MAIN OBSTACLE : One needs a source and detector of neutrinos and the beam has to take two paths between them.
- We can not do what is done in optics : use mirrors and lenses to separate and recombine a beam
- But, we can think of oscillations as doing a split beam experiment in energy space by doing collapses and intermediate adiabatic evolutions.



Split beam two path interferometer

Neutrinos and Polarisation optics

Effect of an arbitrary medium

$$\mathbb{H} = D\mathbb{I} + A\sigma_x + B\sigma_y + C\sigma_z$$

D just gives an overall phase, while A, B, C generate non-trivial optical effects.

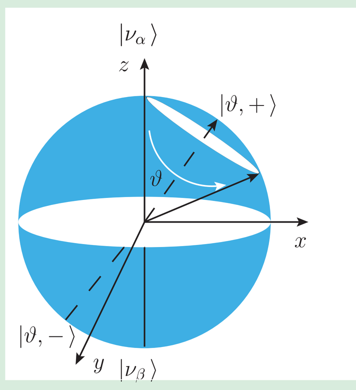
- D and C non-zero : Circular birefringence (Optical activity)
- D and A non-zero : Linear birefringence (Wave plate)
- D and A, B, C non-zero : Elliptic birefringence (Quartz plate)
- Absorptive effects like Dichroism : $\mathbb{H}$ need not be Hermitian

Neutrino oscillations

- Vacuum oscillations : Elliptic birefringence $A = \frac{\omega}{2}\sin 2\theta$, $B = 0$, $C = -\frac{\omega}{2}\cos 2\theta$
- Oscillations in matter : Elliptic birefringence $A = \frac{\omega}{2}\sin 2\theta$, $B = 0$, $C = -\frac{\omega}{2}\cos 2\theta + \frac{1}{2}\sqrt{2}G_F n_e$
- Absorption negligible since neutrinos interact via weak interactions.

Visualization tool - The Poincaré sphere

$$|\psi\rangle = e^{i\eta}\begin{pmatrix} \cos(\vartheta/2)\ e^{-i\phi/2} \\ \sin(\vartheta/2)\ e^{i\phi/2} \end{pmatrix}$$



- $\mathbb{H}_\nu(\theta)$ is real ($x - z$ plane)
- Half angles used : $\vartheta = 2\theta$
- Orthogonal states - antipodal points $|\nu_\alpha\rangle$ and $|\nu_\beta\rangle \Leftrightarrow$ RCP and LCP states
- $|\vartheta,+\rangle$ and $|\vartheta,-\rangle \Leftrightarrow$ EP states
- Oscillation phenomena can be viewed as precession, unitary rotations
- MSW effect $\Rightarrow \theta = \pi/4$ complete swapping of flavors
- NP rotated into SP (about equator with LP states at anti-podal points)

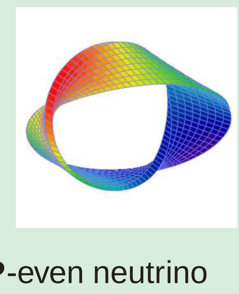
Polarised states in optics have isomorphic connection with the neutrino states

The form of $\mathbb{H}_\nu$ and π anholonomy

$$\mathbb{H}_\nu = \frac{\omega}{2}[-\cos\vartheta\sigma_z + \sin\vartheta\sigma_x]$$

$$|\vartheta,+\rangle = \begin{pmatrix} \cos\vartheta/2 \\ \sin\vartheta/2 \end{pmatrix} |\vartheta,-\rangle = \begin{pmatrix} -\sin\vartheta/2 \\ \cos\vartheta/2 \end{pmatrix}$$

- CP conserved (coeff. of σ_y = 0), Eigenstates lie on a great circle (intersection of $x - z$ plane with Poincaré sphere)
- Eigenstates change sign as ϑ changes from $0 \to 2\pi$,



$$|\vartheta,\pm\rangle = \mp|\vartheta+\pi,\mp\rangle = -|\vartheta+2\pi,\pm\rangle$$
$$= \pm|\vartheta+3\pi,\mp\rangle = |\vartheta+4\pi,\pm\rangle$$

Global structure $\rightsquigarrow$ Möbius band

- Expect a phase of ± 1 (molecular physics) to appear in the CP-even neutrino system. Longuet-Higgins et. al. (1958)

The π phase in the transition probability

Transition probability

Start with flavor states $|\nu_\alpha>$, $|\nu_\alpha> = \nu_{\alpha+}|\vartheta_1,+> + \nu_{\alpha-}|\vartheta_1,-> $ where, $|\vartheta_1,\pm>$ are the eigenstates of $\mathbb{H}_\nu(\vartheta_1) = [(\sin\vartheta_1)\sigma_x + (-\cos\vartheta_1)\sigma_z]$.
Adiabatic evolution of mass states from $|\vartheta_1,\pm>$ to $|\vartheta_2,\pm>$

$$|\vartheta_1,\pm> \ \to \ e^{-i\mathcal{D}_\pm}|\vartheta_2,\pm> \quad \text{with}$$
$$\mathcal{D}_\pm \ \simeq \ \pm\frac{1}{2}\int\sqrt{(\omega\sin\vartheta)^2 + (V_C - \omega\cos\vartheta)^2}dt$$

Amplitude : $\mathcal{A}(\nu_\alpha \to \nu_\beta) = \langle\nu_\beta|\mathcal{U}|\nu_\alpha\rangle$ where

$$\mathcal{U} \ = \ e^{-i\mathcal{D}_+}|\vartheta_2,+>\langle\vartheta_1,+| + e^{-i\mathcal{D}_-}|\vartheta_2,-\rangle\langle\vartheta_1,-| \, .$$

Probability :

$$\begin{aligned}\mathcal{P}(\nu_\alpha\to\nu_\beta) \ = \ &\langle\nu_\alpha|\vartheta_1,+\rangle\langle\vartheta_2,+|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2,+\rangle\langle\vartheta_1,+|\nu_\alpha\rangle \\ &+ \langle\nu_\alpha|\vartheta_1,-\rangle\langle\vartheta_2,-|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2,-\rangle\langle\vartheta_1,-|\nu_\alpha\rangle \\ &+ [\langle\nu_\alpha|\vartheta_1,-\rangle e^{i\mathcal{D}_-}\langle\vartheta_2,-|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2,+\rangle e^{-i\mathcal{D}_+}\langle\vartheta_1,+|\nu_\alpha\rangle + \text{c.c.}]\end{aligned}$$

- cross-terms (upon removing the dynamical phase) are connected to the two path interferometer in energy space
- can be viewed as closed loop quantum collapses with intermediate adiabatic evolutions
- great circle in $x - z$ plane.

Standard expression for probability

$$\begin{aligned}\mathcal{P}(\nu_e\to\nu_\mu) \ = \ &\mathbb{U}^\star_{e+}(\theta_1)\mathbb{U}_{\mu+}(\theta_2)\mathbb{U}^\star_{\mu+}(\theta_2)\mathbb{U}_{e+}(\theta_1) \\ &+ \ \mathbb{U}^\star_{e-}(\theta_1)\mathbb{U}_{\mu-}(\theta_2)\mathbb{U}^\star_{\mu-}(\theta_2)\mathbb{U}_{e-}(\theta_1) \\ &+ [\mathbb{U}^\star_{e-}(\theta_1)e^{i\mathcal{D}_-}\mathbb{U}_{\mu-}(\theta_2)\mathbb{U}^\star_{\mu+}(\theta_2)e^{-i\mathcal{D}_+}\mathbb{U}_{e+}(\theta_1) + \text{c.c.}]\end{aligned}$$

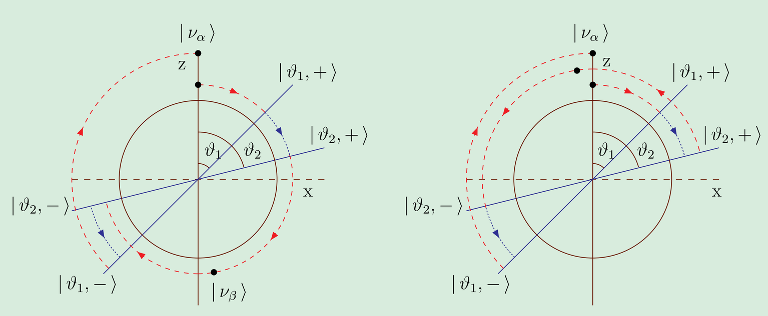
Interference terms as collapses + adiabatic evolutions

Cross-term (upon removing dynamical phase)

$\langle\nu_\alpha|\vartheta_1,-\rangle\langle\vartheta_2,-|\nu_\beta\rangle\langle\nu_\beta|\vartheta_2,+\rangle\langle\vartheta_1,+|\nu_\alpha\rangle \equiv re^{i\beta}$

Appearance terms ($\nu_\alpha \to \nu_\beta$) | Disappearance terms ($\nu_\alpha \to \nu_\alpha$)

$\beta = \pi$ | $\beta = 0$



(a) (b)

$$\mathcal{P}(\nu_e\to\nu_\mu) = \cos^2\theta_1\sin^2\theta_2 + \sin^2\theta_1\cos^2\theta_2 + [2\cos(\mathcal{D}_+ - \mathcal{D}_-)](-\sin\theta_1)\cos\theta_2\sin\theta_2\cos\theta_1$$

topological phase = π

$$\mathcal{P}(\nu_e\to\nu_e) = \cos^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\sin^2\theta_2 + [2\cos(\mathcal{D}_+ - \mathcal{D}_-)]\sin\theta_1\cos\theta_2\sin\theta_2\cos\theta_1$$

topological phase $= 0$, in accord with Unitarity

Conclusions and Outlook

- We show that there is a topological phase in the two flavor neutrino oscillation formulae by using Pancharatnam's ideas and our study leads to first pure geometric interpretation of the phenomenon of oscillations for the specific case of two flavors and CP conserving case.
- The non-trivial phase of π and the anholonomy is linked to encircling of a singular point in ray space.
- We made a direct connection to the π phase anholonomy first found in the context of molecular physics by Longuet-Higgins et. al. in 1958.
- The phase remains irrespective of adiabatic evolution or propagation of neutrinos in vacuum and is a robust quantity.
- It is in-built into the structure of the leptonic mixing matrix. For the 2×2 case, $\mathbb{U}(\theta)$ is a real orthogonal rotation matrix given by,

$$\mathbb{U}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Therefore the standard formalism of oscillation is in fact a realization of the Pancharatnam phase.

- The topological robustness can be destroyed once we invoke CP violation.

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