Degenerate neutrinos and maximal mixing

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Fermion mass puzzle : several facets

- family replication and the resulting proliferation of couplings,
- the observed hierarchy in the fermion masses and mixing angles,
- the origin of CP violation in weak interactions and its absence in strong interactions, and
- the origin of tiny neutrino masses.

Quark mixing ⇝ CKM matrix

$$\mathbb{V}_{CKM} = \mathbb{V}_u \mathbb{V}_d^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$= \begin{pmatrix} 0.97419 & 0.2257 & 0.00359 \\ 0.2256 & 0.97334 & 0.0415 \\ 0.00874 & 0.0407 & 0.999133 \end{pmatrix}$$

Leptonic mixing ⇝ PMNS matrix

$$\mathbb{U}_{PMNS} = \mathbb{U}_l^\dagger \mathbb{U}_\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

$$|\mathbb{U}_{PMNS}|_{3\sigma} = \begin{pmatrix} 0.77-0.86 & 0.50-0.63 & 0.00-0.22 \\ 0.22-0.56 & 0.44-0.73 & 0.57-0.80 \\ 0.21-0.55 & 0.40-0.71 & 0.59-0.82 \end{pmatrix}$$

Fermion mass (MeV)

http://pdg.lbl.gov

u 1:5 to 3:3	c 1270	t 171300
d 3.5 to 6.0	s 105	b 4200
e 0.511	μ 106	τ 1777
ν_e < 0.02	ν_μ < 0.19	ν_τ < 18.2

Our focus is to obtain large mixing and tiny mass naturally in the neutrino sector.

Unresolved issues at the present epoch

Mass pattern

- Normal hierarchy: $\Delta m^2_{31} > 0$
- Inverted hierarchy: $\Delta m^2_{31} < 0$
- Quasi-Degenerate: $m_1 \simeq m_2 \simeq m_3$

Mixing and CP violation

- Why two large and one small
- Dirac-type CP violation

Absolute mass and its origin

- future β decay, $0\nu\beta\beta$ experiments
- Need to invoke beyond SM physics to give masses to neutrinos
- If Majorana, seesaw mechanism can explain tininess

Dirac vs Majorana

- $0\nu\beta\beta$ experiments will be able to tell if neutrinos are Majorana

2 Majorana phases

- very hard but $0\nu\beta\beta$ process is sensitive to them

Neutrino flavour parameters and low energy data

Oscillation data

M. C. Gonzalez-Garcia and M. Maltoni, Phys. Rep. 460, 1 (2008), R. Z. Funchal, talk at ν2008

- 3 mixing angles : $\boxed{\theta_{12} \approx 32^o\ ,\ \theta_{23} \approx 45^o,\ \theta_{13} \leq 10^o \text{ (upper bound)}}$
- 2 mass-squared differences : $\boxed{\Delta m^2_{21} \simeq 7.7 \times 10^{-5} eV^2 \text{ and } |\Delta m^2_{31}| \simeq 2.5 \times 10^{-3} eV^2}$

Non-oscillation data

- β-decay experiments, $m_\beta = \sqrt{\sum_i |\mathbb{U}_{ei}|^2 m_i^2} = \sqrt{c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2}$, $\boxed{m_\beta < 1.8 \text{ eV}}$ (Mainz+Troitsk)
- $0\nu\beta\beta$ experiments (sensitive to Majorana nature & phases), $m_{\beta\beta} = |\sum_i \mathbb{U}_{ei}^2 m_i| = |c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 e^{i\alpha_2} m_2 + s_{13}^2 e^{i\alpha_3} m_3|$, $\boxed{m_{\beta\beta} \sim 0.16 - 0.52(0 - 0.25) \text{ eV}}$ (Heidelberg-Moscow (Cuoricino))
- Cosmology, $\Omega_\nu \propto \sum = \sum_i m_i$, $\boxed{\sum < 1.3 eV}$ (WMAP5)

This work is an effort to

explain the origin of large/maximal mixing in the neutrino sector by exploiting the fact that in most grand unified models, usually there are more than one seesaw mechanisms at work or there are more than one independent sources of neutrino mass. With only one seesaw mechanism, it is possible to explain the smallness of neutrino masses, but there are some problems viz., the mixing angles are small and CKM like. Here we consider more than one seesaws which is a natural feature of most grand unified models and show that we can get rid of this problem. We show that the small mixing can get enhanced to large or maximal by adding the two seesaws in an appropriate way. This is intimately connected with near degenerate neutrino mass pattern. Alternately, we can decompose the quasi-degenerate mass pattern in to sum of hierarchial and inverse hierarchial patterns with small mixings. The left-right symmetric model where the type I and the type II seesaw mechanisms are related by the same Yukawa provides a framework where small mixings can be converted to large mixing angles, for degenerate neutrinos. With three generations, we show that either one or all three mixing angles can become large, which is not desired. We propose a way to obtain two large and one small mixing angle if either one or both the sub matrices contain large mixing.

Illustration of degeneracy induced large mixing in the two generation case

Consider

$$\mathbb{M}_\nu = \mathbb{M}_\nu^{(1)} + \mathbb{M}_\nu^{(2)}$$

where $\mathbb{M}_\nu^{(1)}$ may originate from Type I Seesaw and $\mathbb{M}_\nu^{(2)}$ may come from Type II Seesaw in a model like $SO(10)$ where both these are simultaneously present

$$\mathbb{M}_\nu = \begin{pmatrix} m_{ee}^{(1)} & m_{e\mu}^{(1)} \\ m_{e\mu}^{(1)} & m_{\mu\mu}^{(1)} \end{pmatrix} + \begin{pmatrix} m_{ee}^{(2)} & m_{e\mu}^{(2)} \\ m_{e\mu}^{(2)} & m_{\mu\mu}^{(2)} \end{pmatrix}$$

The mixing is given by

$$\tan 2\theta = \frac{2(m_{e\mu}^{(1)} + m_{e\mu}^{(2)})}{m_{\mu\mu}^{(2)} + m_{\mu\mu}^{(1)} - m_{ee}^{(2)} - m_{ee}^{(1)}} = \tan 2\theta^{(1)} \frac{1}{1+d} + \tan 2\theta^{(2)} \frac{d}{1+d}$$

where $d = \dfrac{m_{\mu\mu}^{(2)} - m_{ee}^{(2)}}{m_{\mu\mu}^{(1)} - m_{ee}^{(1)}}$

If θ^i are small, large mixing requires $\boxed{d = -1}$

Small $\theta^{(i)}$: $2m_{e\mu}^{(i)} << (|m_{\mu\mu}^{(i)} - m_{ee}^{(i)}|)$ and $m_{\mu\mu}^{(i)} \neq m_{ee}^{(i)}$

$$\Delta m^2 = (m_{ee}^{(1)} + m_{\mu\mu}^{(1)} + m_{ee}^{(2)} + m_{\mu\mu}^{(2)}) \times \sqrt{(m_{\mu\mu}^{(1)} + m_{\mu\mu}^{(2)} - m_{ee}^{(1)} - m_{ee}^{(2)})^2 + 4(m_{e\mu}^{(1)} + m_{e\mu}^{(2)})^2}$$

Assuming at least one diagonal entry in each $\mathbb{M}_\nu^{(i)}$ to be large the condition $\boxed{d = -1}$ leads to sub-cases:

- (A) $m_{\mu\mu}^{(2)} = -m_{\mu\mu}^{(1)}$
- (B) $m_{ee}^{(2)} = -m_{ee}^{(1)}$
- (C) $m_{ee}^{(2)} = m_{\mu\mu}^{(1)}$ or $m_{\mu\mu}^{(2)} = m_{ee}^{(1)}$

Case (A) : $m_{\mu\mu}^{(2)}$ and $m_{\mu\mu}^{(1)}$ are large

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} z & x \\ x & (1+z') \end{pmatrix} + m_2 \begin{pmatrix} 0 & y \\ y & -1 \end{pmatrix}$$

Large mixing condition: $m_1 \simeq m_2 \simeq m$

$$\mathbb{M}_\nu = m \begin{pmatrix} z & x+y \\ x+y & z' \end{pmatrix}$$

Note $\mathcal{O}(m)$ element vanishes and small entries appear in the total mass matrix. $d = -1 \Rightarrow z = z'$ while (A) $\Rightarrow z' = 0$.

$$\Delta m^2 = m^2(z+z')\sqrt{4(x+y)^2 + (z-z')^2}$$

$\tan 2\theta = \dfrac{2(x+y)}{z-z'}$ large mixing possible when $x + y >> z$ for $z' \to 0$

or $x, y \sim z, z'$ when $z \simeq z'$ & hierarchial pattern emerges

Case (B): $m_{ee}^{(2)}$ and $m_{ee}^{(1)}$ are large similar to case (A); need strong cancellations of dominant elements.

Case (C) : For example, let $m_{\mu\mu}^{(2)} = m_{ee}^{(1)}$

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} 0 & x \\ x & 1 \end{pmatrix} + m_2 \begin{pmatrix} 1 & -y \\ -y & 0 \end{pmatrix}$$

$$\mathbb{M}_\nu = \begin{pmatrix} m_2 & m_1 x - m_2 y \\ m_1 x - m_2 y & m_1 \end{pmatrix}$$

Individual mixings are small. But, $\tan 2\theta = \dfrac{2(m_1 x - m_2 y)}{m_1 - m_2}$ can be enhanced if m_1 and m_2 both have same sign.

$|m_1 - m_2| < 2(m_1 x - m_2 y)$ for large mixing.

Spectrum hints towards a quasi-degenerate pattern.

$\Delta m^2 = 4m\epsilon$ in limit $m_1 \sim m_2 \sim m$ and $\epsilon = m(x-y)$

Finally, if two diagonal entries are large in individual matrices, we can have a new class of solutions

Case (C1):

$$\mathbb{M}_\nu = m_1 \begin{pmatrix} 1+\rho & x \\ x & 1 \end{pmatrix} + m_2 \begin{pmatrix} 1 & x' \\ x' & 1+\rho' \end{pmatrix}$$

$$\mathbb{M}_\nu = \begin{pmatrix} m_2 + m_1(1+\rho) & m_1 x + m_2 x' \\ m_1 x + m_2 x' & m_1 + m_2(1+\rho') \end{pmatrix}$$

where x, x' are small and ρ, ρ' are such that $2x/\rho << 1$ and $2x'/\rho' << 1$ to keep mixing small in $\mathbb{M}_\nu^i$. This would mean hierarchy between the elements in $\mathbb{M}_\nu^i$. Qualitatively this could form a new class of solutions with each sub-matrices forming a quasi-degenerate pair with small mixing. However mixing $\tan 2\theta \simeq (x + x')/(\rho' - \rho)$ would remain small as $x, x' << \rho, \rho'$ unless $\rho = \rho'$. But, then this will fall in class (C) where both the conditions on diagonal elements are satisfied. To sum up, the sum of 2 matrices with small mixing angles would naturally lead to a degenerate spectrum with maximal/large mixing provided there are no cancellations of large eigenvalues of individual matrices. The individual matrices could have the form (a) NH + IH (Case C) or (b) quasi-degenerate themselves but with small mixing (Case C1).

Thus to convert small mixing into maximal/large, we need a pair of (quasi-) degenerate eigenvalues with same CP parity, ordered oppositely in the sub matrices. This is “degeneracy induced large mixing”.

Decomposition of degenerate spectrum :

Degenerate spectrum with large mixing can be decomposed into two matrices with small mixings.

This decomposition is more general irrespective of mixing.

Zeroth order textures

Mixing ⇒ $\mathbb{M}_{diag}$	Small $\mathbb{X}_\epsilon$	Maximal $\mathbb{X}_M$
NH $\mathbb{A}$: Diag[0,1]	$\begin{pmatrix} 0 & \epsilon \\ \epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$
IH $\mathbb{B}$: Diag[1,0]	$\begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 0 \end{pmatrix}$	$\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$
Degenerate $\mathbb{C}_0$: Diag[1,1]	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbb{C}_1$: Diag[-1,1]	$\begin{pmatrix} -1 & 2\epsilon \\ 2\epsilon & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbb{C}_2$: Diag[1,-1]	$\begin{pmatrix} 1 & -2\epsilon \\ -2\epsilon & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Note : $\mathbb{C}_0 = \mathbb{A} + \mathbb{B}$; $\mathbb{C}_1 = \mathbb{A} - \mathbb{B}$; $\mathbb{C}_2 = \mathbb{B} - \mathbb{A}$.

Can be incorporated easily in models of neutrino masses.

Model considerations and extension to the three generation case

Single (Type I) Seesaw

Minkowski (1977), Gell-Mann, Glashow, Mohapatra, Ramond, Senjanovic, Slanski, Yanagida (1979/1980), Valle

$$-\mathbb{M}^I = v^2 \begin{pmatrix} h^D_{ee} & h^D_{\mu e} \\ h^D_{e\mu} & h^D_{\mu\mu} \end{pmatrix} \begin{pmatrix} 1/M_{R1} & 0 \\ 0 & 1/M_{R2} \end{pmatrix} \begin{pmatrix} h^D_{ee} & h^D_{e\mu} \\ h^D_{\mu e} & h^D_{\mu\mu} \end{pmatrix}$$
$$= m_1 \begin{pmatrix} (h^D_{ee})^2 & h^D_{ee} h^D_{e\mu} \\ h^D_{ee} h^D_{e\mu} & (h^D_{e\mu})^2 \end{pmatrix} + m_2 \begin{pmatrix} (h^D_{\mu e})^2 & h^D_{\mu e} h^D_{\mu\mu} \\ h^D_{\mu e} h^D_{\mu\mu} & (h^D_{\mu\mu})^2 \end{pmatrix}$$

Parameter region for quasi-degeneracy and large mixing : If $M_{R1} = M_{R2}$ the Yukawa parameters can be $h^D_{e\mu} \sim \mathcal{O}(1), h^D_{ee} \sim x, h^D_{\mu\mu} \sim -y, h^D_{\mu e} \sim \mathcal{O}(1)$ or $h^D_{ee} \sim \mathcal{O}(1), h^D_{e\mu} \sim -x, h^D_{\mu e} \sim y, h^D_{\mu\mu} \sim \mathcal{O}(1)$

Thus each RH neutrino couples with LH neutrinos with small mixing but the total mass matrix ensures maximal mixing.

Two Seesaws (Type I+II)

Lazarides, Magg, Mohapatra, Senjanovic, Shafi, Wetterich (1981)

Left-Right symmetric model

Joshipura (1994,1995), Akhmedov and Frigerio (2006), ...

$$\mathcal{L}_M = -\frac{f}{2}\left(\overline{\nu_L^c}\nu_L \Delta_L^0 + \overline{\nu_R^c}\nu_R \Delta_R^0\right) + h.c.$$

where $\Delta_{L(R)}$ is the triplet Higgs field

$$\mathcal{L}_D = -Y \overline{\nu}_L \nu_R \phi^0 + h.c.$$

In the limit where $v_R \gg v$, total mass matrix is

$$\mathbb{M}_\nu = f v_L - \frac{v^2}{v_R} Y f^{-1} Y^T \rightarrow Type\ II(NH) + Type\ I(IH)$$

Can choose the Yukawa textures as

$$f = \begin{pmatrix} 0 & x \\ x & 1 \end{pmatrix} \quad , \quad Y = \begin{pmatrix} 1 & y \\ y & 0 \end{pmatrix}$$

leading to

$$\mathbb{M}_\nu = \begin{pmatrix} 0 & m_1 x \\ m_1 x & m_1 \end{pmatrix} + \frac{m_2}{x^2} \begin{pmatrix} 1-2xy & y(1-xy) \\ y(1-xy) & y^2 \end{pmatrix}$$
$$= \frac{1}{x^2} \begin{pmatrix} m_2(1-2xy) & m_1 x^3 + m_2 y(1-xy) \\ m_1 x^3 + m_2 y(1-xy) & m_1 x^2 + m_2 y^2 \end{pmatrix}$$

where $m_1 = v_L$ and $m_2 = v^2/v_R$.

Mixing angle,

$$\tan 2\theta = \frac{2(m_1 x^3 + m_2 y(1-xy))}{m_1 x^2 + m_2 y^2 - m_2(1-2xy)}$$

Degeneracy requirement $m_1 x^2 \simeq m_2 \longrightarrow$ large mixing

Effect of radiative corrections,

$$\tan 2\theta = \frac{2((m_1 x^3 + m_2 y(1-xy))(1+\delta)}{(1+\delta)^2 m_1 x^2 + m_2 y^2 - m_2(1-2xy)}$$

Now, $m_1 x^2 (1+\delta)^2 \approx m_2$ for large mixing and keeping degeneracy stable. Of course, the splitting of the degeneracy can come from the radiative effects.

Three generation case: two possibilities

(i) Two Seesaw mechanisms or two sources of neutrino masses: Total mass matrix (with 3 small mixings in the two sub matrices)

$$\mathbb{M}_\nu = \begin{pmatrix} m_1^{(1)} + m_1^{(2)} & (m_2^{(1)} - m_1^{(1)})\epsilon_{12}^{(1)} + (m_2^{(2)} - m_1^{(2)})\epsilon_{12}^{(2)} & (m_3^{(1)} - m_1^{(1)})\epsilon_{13}^{(1)} + (m_2^{(2)} - m_1^{(2)})\epsilon_{13}^{(2)} \\ \star & m_2^{(1)} + m_2^{(2)} & (m_3^{(1)} - m_2^{(1)})\epsilon_{23}^{(1)} + (m_3^{(2)} - m_2^{(2)})\epsilon_{23}^{(2)} \\ \star & \star & m_3^{(1)} + m_3^{(2)} \end{pmatrix}$$

Constraint - θ_{13} must not be large. We can generate only one large mixing angle in this case when there are two sub matrices. Assuming that one of the matrices enhances the small mixing via degeneracy condition, while the other generates large mixing, we propose textures for this. Here too we can decomposition degenerate spectrum into 2 sub matrices

Mixing ⇒	Small	Single maximal	Bimaximal	Tribimaximal
$\mathbb{A}$: Diag[0,0,1]	$\begin{pmatrix} 0 & 0 & \epsilon_{13} \\ 0 & 0 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac12 & \frac12 \\ 0 & \frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac12 & \frac12 \\ 0 & \frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac12 & \frac12 \\ 0 & \frac12 & \frac12 \end{pmatrix}$
$\mathbb{B}_1$: Diag[1,-1,0]	$\begin{pmatrix} 1 & -2\epsilon_{12} & -\epsilon_{13} \\ -2\epsilon_{12} & -1 & \epsilon_{23} \\ -\epsilon_{13} & \epsilon_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac12 & \frac12 \\ 0 & \frac12 & -\frac12 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & 0 & 0 \\ \frac{1}{\sqrt2} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac13 & -\frac23 & \frac23 \\ -\frac23 & -\frac16 & \frac16 \\ \frac23 & \frac16 & -\frac16 \end{pmatrix}$
$\mathbb{B}_2$: Diag[1,1,0]	$\begin{pmatrix} 1 & 0 & -\epsilon_{13} \\ 0 & 1 & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac12 & -\frac12 \\ 0 & -\frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac12 & -\frac12 \\ 0 & -\frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac12 & -\frac12 \\ 0 & -\frac12 & \frac12 \end{pmatrix}$
$\mathbb{C}_0$: Diag[1,1,1]	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\mathbb{C}_1$: Diag[-1,1,1]	$\begin{pmatrix} -1 & 2\epsilon_{12} & 2\epsilon_{13} \\ 2\epsilon_{12} & 1 & 0 \\ 2\epsilon_{13} & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac12 & \frac12 \\ -\frac{1}{\sqrt2} & \frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} -\frac13 & \frac23 & -\frac23 \\ \frac23 & \frac23 & \frac13 \\ -\frac23 & \frac13 & \frac23 \end{pmatrix}$
$\mathbb{C}_2$: Diag[1,-1,1]	$\begin{pmatrix} 1 & -2\epsilon_{12} & 0 \\ -2\epsilon_{12} & -1 & 2\epsilon_{23} \\ 0 & 2\epsilon_{23} & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt2} & \frac12 & \frac12 \\ \frac{1}{\sqrt2} & \frac12 & \frac12 \end{pmatrix}$	$\begin{pmatrix} \frac13 & -\frac23 & \frac23 \\ -\frac23 & \frac13 & \frac23 \\ \frac23 & \frac23 & \frac13 \end{pmatrix}$
$\mathbb{C}_3$: Diag[1,1,-1]	$\begin{pmatrix} 1 & 0 & -2\epsilon_{13} \\ 0 & 1 & -2\epsilon_{23} \\ -2\epsilon_{13} & -2\epsilon_{23} & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

(ii) Three Seesaw mechanisms : 3 Maximal Mixing Angles Cabibbo and Wolfenstein